PART 1. Q^4 IS NOT IN $\{P\}$.

In this case $Q^8=P^{8\lambda}$, and G_m is generated by P and Q, since there are 2^m distinct operators of the form $Q^\alpha P^\beta$ ($\alpha=0\ldots7$, $\beta=0\ldots2^{m-3}-1$). Also $G_{m-3}=\{P\}$, and G_{m-2} is generated by P and some other operator of G_m, $Q^\alpha P^\beta$. Hence G_{m-2} contains Q^α. So $G_{m-2}=\{P, Q^\alpha\}=\{P, Q^4\}$. Likewise $G_{m-1}=\{P, Q^2\}$.

In G_{m-2}, $Q^{-4}PQ^4=P^{\omega+2^{m-4}\kappa}$; in G_{m-1}, $Q^{-2}PQ^2=Q^{4a}P^\beta$, and in G_m, $Q^{-1}PQ=Q^{2a}P^b$. Three cases arise: (A) $a=a=0$; (B) $a=0$, $a=1, 2, 3$; (C) $a=1$, $a=1, 2, 3$. We shall subdivide these cases where necessary, using $A_1, A_2, \ldots$ to distinguish the 1st, 2nd, ... subcases of A, etc.

(A) $a=a=0$. Here $\{P\}$ is self-conjugate in G_m.

$$Q^{-1}PQ=P^b=P^{\omega_1+2^{m-6}b_1}\ (\omega_1=\pm1,\ b_1=0\ldots7)\ldots(1).$$

Let $b_1=2^nb_2$, where b_2 is odd and $n=0\ldots3$. In general, $(R^zQ^yP^xR^v\ldots)^s$ will be represented by $[z, y, x, v\ldots]^s$; also $\frac{1+(-1)^{s-1}}{2}$ by θ_s, and $\frac{1+(-1)^s}{2}$ by ϕ_s. From (1),

$$[0, -y, x, 0, y]=[0, 0, x\ (\omega_1+2^{m-6+n}b_2)^y]\ldots(2).$$

$$[0, y, x]^2=[0, 2y, 2x\{\frac{1+(\omega)^y}{2}+(\omega)^{y-1}2^{m-7+n}b_2y\}]\ldots(3).$$

Let $Q'=QP^x$; then $Q'^8=1$, if x be chosen to satisfy

$$\lambda+x(1+\omega_1)/2+2^{m-7+n}b_2x\equiv0\ (\mathrm{mod}\ 2^{m-6})\ldots(4),$$

where for $\omega_1=-1$, $\lambda=2^{m-7}\lambda_1$. This is always possible for $\omega_1=1$, except when $m=7$; b_1 and λ odd, when $Q^8=P^8$; and for $\omega_1=-1$, when b_1 is odd. For $\omega_1=-1$, and b_1 even, $Q'^8=P^{2^{m-4}\lambda_1}$; $\lambda_1=0, 1$. In this last case,

$$[0, y, x]^s=[0, sy, x\ (s\phi_y+\theta_{sy})+2^{m-7+n}b_2xy\{s-s^2\phi_y-(2s-1)\theta_{sy}\}]\ldots(5),$$

and the groups, where $n=2$, correspond to those where $n=3$, for $\lambda_1=1$. The correspondence is given by $C=\begin{bmatrix}Q^4P, & Q\\ P, & Q\end{bmatrix}$.

Let $Q'=Q^y$; then $Q'^{-1}PQ'=P^{\omega_1+2^{m-6+n}}$, if y be chosen to satisfy

$$b_2y+\frac{y(y-1)}{2}2^{m-6+n}b_2^2+\ldots\equiv1\ (\mathrm{mod}\ 2^{3-n})\ldots(6),$$

which choice is always possible. So in (A) there are ten types. They are given by the following defining relations:

$Q^{-1}PQ=P^{\omega_1+2^{m-6+n}}$, $Q^8=P^{2^{m-4}\lambda_1}$, $P^{2^{m-3}}=1$; $\lambda_1=0$; $\omega_1=\pm1$, $n=1, 2, 3$;

$m>7$, $\omega_1=\pm1$, $n=0$. $m=7$, $\omega_1=-1$, $n=0$; $\lambda_1=1$, $\omega_1=-1$, $n=1, 3$.

$Q^{-1}PQ=P^{1+2}$, $Q^8=P^8$, $P^{2^4}=1$.

(B) $a=0$, $a=1, 2, 3$.

$$Q^{-1}PQ=Q^{2^a}P^{b}\ldots(1),$$

$$Q^{-2}PQ^2=P^{\omega_1+2^{m-5}\beta_1}\ (\omega_1=\pm 1,\ \beta_1=0\ldots 3)\ldots(2).$$

From (2), $[0, -2y_1, x, 0, 2y_1]=[0, 0, x(\omega_1+2^{m-5}\beta_1)^{y_1}]\ldots(3)$. Hence

$$[0, 2y_1, x]^s=[0, 2sy_1, \theta_s x\{1+\omega_1 2^{m-5}\beta_1 y_1(s-\theta_s)\}+(s-\theta_s)\{\frac{1+(\omega_1)^{y_1}}{2}$$

$$+(\omega_1)^{y_1-1}2^{m-6}\beta_1 y_1]x]\ldots(4).$$

Transform (1) by Q and place equal to (2); raise to the power 8λ. There result $b=1+2b_1$, and the congruences

$$a(1+b_1)\equiv 0 \pmod 2\ldots(5);$$

$$\lambda[4a\lambda-1+2b_1+(\omega_1)^a(1+2b_1)]\equiv 0 \pmod{2^{m-5}}\ldots(6),$$

$$4a\lambda(1+b_1)+(1+2b_1)[1+b_1(1+(\omega_1)^a)]+2^{m-5}\beta_1[(\omega_1)^{a-1}ab_1+1]$$

$$\equiv\omega_1 \pmod{2^{m-3}}\ldots(7).$$

For $\omega_1=-1$, a and b are odd by (8). Hence P is of an order lower than 2^{m-3}, and there are no groups in this case.

For $\omega_1=1$, from (1) and (4),

$$[0, -y, x, 0, y]=[0, 2ax\theta_y, x\{1+2^{m-6}\beta_1(y-\theta_y)\}\{1+2\theta_y[b_1+2^{m-7}a\beta_1(1+2b_1)]\}$$

$$+2^{m-6}a\beta_1\theta_y\{2\lambda x(y-1)+\theta_x(1+2b_1)(2x-3)\}]\ldots(8),$$

and $[0, y, x]^2=[0, 2(y+ax\theta_y), x+x\{1+2^{m-6}\beta_1(y-\theta_y)\}\{1+2\theta_y[b_1+2^{m-7}a\beta_1$

$$\times(1+2b_1)]\}+2^{m-6}a\beta_1\theta_y\{2\lambda x(y-1)+\theta_x(1+2b_1)(2x-3)\}]\ldots(9).$$

(B_1) $a=2$.

(a) $b_1=2^{m-7}b_2$; (b) $b_1=-1+2^{m-7}b_2$; $(b_1=0\ldots 7)$. Let $Q'=QP^{-\lambda}$. Then, for (a), $Q'^8=1$; for (b), $Q'^8=P^{2^{m-4}\lambda_1}$ $(\lambda_1=0,\ 1)$. $(b)\sim(A)$ with $C=\begin{bmatrix} Q^2P, & Q \\ P, & Q\end{bmatrix}$. $(a)\sim(a_1)$, (a_2), where $b_2=\beta_1=0,\ 2$, respectively; and (a_3) where $b_2=\beta_1=1$, for $m>8$; and $b_2=3$, $\beta_1=1$, for $m=8$. $C=\begin{bmatrix} P, & Q^{y'}Q^{2^{m-7}x_1'} \\ P, & Q\end{bmatrix}$, where $y'=1+2y'_1$ and y'_1 and x'_1 are determined to satisfy the congruences $\beta_1(1+2y'_1)\equiv\beta'_1 \pmod 4$; $\lambda_1+x'_1\equiv\lambda'_1 \pmod 2$; and $2y'_1(\beta_1-2\lambda_1)-2x'_1\equiv b'_2-b_2 \pmod 8$. Of these (a_3) for $m=8$, and (a_2) for $m=7\sim(A)$ with $C=\begin{bmatrix} QP, & QP^4 \\ P, & Q\end{bmatrix}$ and $\begin{bmatrix} QP, & QP^2 \\ P, & Q\end{bmatrix}$, respectively. Hence there are three types for $m>8$, two for $m=8$, and one for $m=7$.

(B_2) $a=1, 3$, $m>7$. From (5), $b_1=1+2b_2$. Let $Q'=QP^{-\lambda}$ for λ even. Then $Q'^8=P^{2^{m-4}\lambda_1}$ $(\lambda_1=0,\ 1)$. Let $Q'=Q^y$ (y odd), for λ odd. Then $Q'^8=P^8$, if y be determined to satisfy $\lambda y\equiv 1 \pmod{2^{m-6}}$.

(a) For $\lambda=2^{m-7}\lambda_1$, from (7) $b_2=-1+2^{m-7}b_3$ ($b_3=0\ldots3$). (a)$\sim$the seven types $a=1$; $b_3=\beta_1=0,\ 1,\ 2$, $\lambda_1=0,\ 1$; $b_3=2$, $\beta_1=0$, $\lambda=0$; with $C=\begin{bmatrix} Q^{2y_1}P^x, & QP^{2x_1'} \\ P, & Q \end{bmatrix}$, where $x=1+2x_1$, and the variables satisfy $a_1-a_1'\equiv x_1+y_1+x_1'$ (mod 2); $\beta_1(1+2x_1')\equiv\beta_1'$ (mod 4); and $\lambda_1\{a_1-a_1'+x_1+y_1-x_1'+2[a_1(x_1-x_1')-a_1'x_1']\}-\beta_1\{1-x_1-x_1'-b_3'y_1+2[a_1(x_1+x_1'+1)+x_1(1+y_1)+a_1'x_1']\}+(b_3-b_3')(1+2x_1)+2b_3x_1'\equiv0$ (mod 4).

(b) For $\lambda=1$, from (6), $b_2=-1-a_1+2^{m-8}b_3$, ($b_3=0\ldots7$; $a=1+2a_1$). (b)$\sim$the seven types $a=1$; $b_3=0$, $\beta_1=0,\ 2$; $a=3$, $b_3=\beta_1=0,\ 1,\ 2$; $m>8$, $a=1$; $m=8$, $a=3$; $b_3=1$, $\beta_1=0, 2$; with $C=\begin{bmatrix} P, & QP^{2^{m-6}x_1'} \\ P, & Q \end{bmatrix}$, where x_1' satisfies $b_3'-b_3+2ax_1'\equiv0$ (mod 8),* except for β_1 odd, where $C=\begin{bmatrix} P^{\beta_1'}, & QP^{2(\beta_1'-1)+2^{m-6}x_1'} \\ P, & Q \end{bmatrix}$ where x_1' satisfies $b_3'\beta_1'-b_3+2ax_1'\equiv0$ (mod 8).

(B_3) $a=1,\ 3$; $m=7$. (B_3), ($a=3$)$\sim$(B_3), ($a=1$); with $C=\begin{bmatrix} P^3, & Q \\ P, & Q \end{bmatrix}$.

(a) $\lambda=0$. There are four types $b_2=1,\ 3$, $\beta_1=0,\ 2$.

(b) $\lambda=1$. (b)$\sim$the five types $b_2=0,\ 1$, $\beta_1=0,\ 2$, and $b_2=0$, $\beta_1=1$; with $C=\begin{bmatrix} P, & QP^{4x_1'} \\ P, & Q \end{bmatrix}$ where x_1' satisfies $b_2'-b_2+2ax_1'\equiv0$ (mod 4), except for β_1 odd, where $C=\begin{bmatrix} P^x, & Q^{y'}P^{2x_1'} \\ P, & Q \end{bmatrix}$, where x and y' are odd and the variables satisfy $x_1+x_1y_1'+x_1'y_1'-b_3'\equiv0$ (mod 2); $\beta_1'\equiv y'+2x_1'$ (mod 4); and $x_1+x_1'+y_1'\equiv0$ (mod 2).

Hence in (B) there are seventeen types for $m>8$, sixteen for $m=8$, ten for $m=7$, viz:

$Q^{-2}PQ^2=P^{1+2^{m-5}\beta_1}$, $P^{2^{m-3}}=1$.

$Q^{-1}PQ=Q^4P^{1+2^{m-6}b_2}$, $Q^8=1$; $m>8$; $b_2=\beta_1=0,\ 1,\ 2$; $m=8$, $b_2=\beta_1=0,\ 2$; $m=7$, $b_2=\beta_1=0$.

$Q^{-1}PQ=Q^2P^{-1+2^{m-4}b_3}$, $Q^8=1$; $m>7$, $b_3=1$, $\beta_1=0$; $m=7$, $b_3=0,\ 1$, $\beta_1=0, 2$.

$Q^{-1}PQ=Q^2P^{-1+2^{m-5}b_3}$, $Q^8=P^{2^{m-4}\lambda_1}$; $m>7$; $b_3=\beta_1=0,\ 1,\ 2$; $\lambda_1=0,\ 1$; $m=7$, $b_3=1$, $\beta_1=0,\ 1,\ 2$, $\lambda_1=1$.

$Q^{-1}PQ=Q^{2(1+2a_1)}P^{-1-4a_1+2^{m-6}b_3}$, $Q^8=P^8$; $a_1=b_3=0$, $\beta_1=0,\ 2$; $m>8$, $a_1=0$, $m=8$, $a_1=1$, $b_3=1$, $\beta_1=0,\ 2$; $m=7$, $a_1=1$, $b_3=\beta_3=0,\ 1,\ 2$.

(C) $a=1$, $a=1,\ 2,\ 3$.

$$Q^{-1}PQ=Q^{2^a}P^b\ldots(1),$$

$$Q^{-2}PQ^2=Q^4P^\beta\ldots(2),$$

$$Q^{-4}PQ^4=P^{\omega+2^{m-4}\kappa}\ldots(3).$$

---

*For $m=8$, b_3 odd and $a=3$, x_1' satisfies $b_3'-b_3+2x_1'\equiv0$ (mod 8).

From (3), $[0, -4y_1, x, 0, 4y_1] = [0, 0, x(\omega + 2^{m-4}\kappa)^{y_1}]$...(4), and

$$[0, 4y_1, x]^s = [0, 4sy_1, (s-\theta_s)\{\frac{1+(\omega+2^{m-4}\kappa)^{y_1}}{2}\}x + \theta_s x]...(5).$$

Now $\omega = 1$; for when $\omega = -1$, (2) raised to the second power shows that P is of an order lower than 2^{m-3}. Also a is odd; for if $a=2$, (1) transformed by Q and placed equal to (2) shows that b is even, which makes P of an order lower than 2^{m-3}. Next transform (2) by Q^2; $\beta = 1 + 2\beta_1$. From (5) and (2),

$$[0, -2y_1, x, 0, 2y_1] = [0, 4x\theta_{y_1}, x(1+2\beta_1\theta_{y_1}) + 2^{m-5}\kappa(xy_1 - \theta_{xy_1})]...(6).$$

Hence,

$$[0, 2y_1, x]^s = [0, 2sy_1 + 2(s-\theta_s)x\theta_{y_1}, sx + 2(s-\theta_s)\{\beta_2 x\theta_{y_1} + 2^{m-7}\kappa(xy_1 - \theta_{xy_1})\}].(7).$$

Square (1) by (7); $b = 1 + 2b_1$, $\beta_1 = 2\beta_2$. From (1) and (7),

$$[0, -y, x, 0, y] = [0, 4x\theta_{y_1} + 2\theta_y\{ax + b(x-\theta_x)\}, 16\lambda\beta_2 x\theta_{yy_1}(1 + a_1 + b_1) + \{x(1 + 4\beta_2\theta_{y_1}) + 2^{m-5}\kappa(xy_1 - \theta_{xy_1})\}\{1 + 2\theta_y(b_1 + \beta_2 + 2b_1\beta_2)\} + 2\theta_y\{-\theta_x\beta_2(1+2b_1) + 2^{m-6}\kappa(a_1 + b_1)(x-\theta_x)\}]...(8).$$

Hence,

$$[0, y, x]^2 = [0, 4x\theta_{y_1} + 2\{y + \theta_y[ax + b(x-\theta_x)]\}, 16\lambda\beta_2\theta_{yy_1}x(1 + a_1 + b_1) + x + x(1 + 4\beta_2\theta_{y_1})\{1 + 2\theta_y(b_1 + \beta_2 + 2\beta_2 b_1)\} + 2^{m-5}\kappa(xy_1 - \theta_{xy_1}) + 2\theta_y\{2^{m-6}\kappa(a_1 + b_1)(x-\theta_x) - \beta_2\theta_x(1+2b_1)\}]...(9).$$

Now transform (1) and (2) by Q^2 and Q, respectively; then raise them to the power 8λ. There result:

$$\beta_2 = -\lambda + 2^{m-7}\beta_3, \ (\beta_0 = 0...3); \ \beta_3 \equiv \kappa \pmod 2 ...(10);$$

$$\lambda\{\lambda(1+2a_1) + b_1\} \equiv 0 \pmod{2^{m-7}}...(11), \text{ and}$$

$$(1+b_1)[\lambda(1+2a_1) + b_1] + 2^{m-7}[\beta_3(b_1 - 1) + 2\kappa b_1 a_1] \equiv 0 \pmod{2^{m-5}}...(12).$$

(C_1) $b = 2b_2$, $m > 7$. From (11), $\lambda = 2\lambda_1$. Let $Q' = QP^{2x_1}$; then $Q'^8 = 1$, where x_1 satisfies $\lambda_1(1+2ax_1) + x_1(1+2b_2) \equiv 0 \pmod{2^{m-5}}$. From (12), $b_1 = 2^{m-7}b_3$ ($b_3 = 0...7$). Thus (C_1) $\sim$ the four types $a = 1, 3$; $\kappa = \beta_3 = b_3 = 0, 1$; with $C = \begin{bmatrix} P, & QP^{2^{m-6}x_1'} \\ P, & Q \end{bmatrix}$, where x_1' satisfies $\beta_3 - \beta_3' + 2x_1' \equiv 0 \pmod 4$, and $b_3 - b_3' \equiv 2a'x_1' \pmod 8$.

(C_2) $b = 1 + 2b_2$, $m > 7$.

(a) For λ even, from (12), $b_1 = -1 + 2^{m-6}b_3$ ($b_3 = 0...3$). From (11), $\lambda = 2^{m-7}\lambda_2$ ($\lambda_2 = 0, 1$). Hence (a) $\sim$ the fourteen types $\lambda_2 = 0, 1$; $\kappa = \beta_3 = b_3 = 0, 1$, $a = 1$; $\kappa = \beta_3 = 0, 1$, $b_3 = 0$, $a = 3$; $\lambda_2 = \kappa = 0$, $\beta_3 = 0, 2$, $b_3 = 2$, $a = 1, 3$; $\lambda_2 = \kappa = 1$, $\beta_3 = 3$; $b_3 = a = 1$; $b_3 = 0$, $a = 3$. The isomorphism is given by $C = \begin{bmatrix} Q^{2y_1}P^x, & QP^{2x_1} \\ P, & Q \end{bmatrix}$ where $x = 1 + 2x_1$, and the variables satisfy $(\beta_3 - \beta_3')x + 2[\beta_3 x_1' + \lambda_2 x_1' + \kappa x_1] \equiv 0 \pmod 4$, and $\lambda_2\{2a_1x_1 + x_1 + y_1 - x_1'\} - \kappa\{2(a_1 - 1)(x_1 + x_1') - (y_1 - \theta_{y_1})\} - \beta_3\{x_1 - a'x_1' - \theta_{y_1}\} - 2b_3x_1' + (b_3 - b_3')x \equiv 0 \pmod 4$.

(b) For λ odd, from (11) $b_1 \equiv -1-2a_1+2^{m-7}b_3$. Let $Q'=Q^y$ (y odd). Then $Q'^8=P^8$ where y satisfies $\lambda_y \equiv 1 \pmod{2^{m-6}}$. Thus ($b$)$\sim$the seven types $\kappa=\beta_3=0$; $b_3=0,\ 2,\ a=1,\ 3$; $b_3=1,\ a=1$;* $\kappa=\beta_3=1,\ b_3=1,\ 3$; with $C=\begin{bmatrix} P, & QP^{2^{m-6}x_1} \\ P, & Q \end{bmatrix}$, where x_1' satisfies $\beta_3-\beta_3'+2x_1' \equiv 0 \pmod 4$, and $b_3-b_3'-2x_1'(1+2a_1) \equiv 0 \pmod 8$;† except for $a=1$, $\kappa=0$; $\beta_3=0$, $b_3=3+4b_4$;‡ $\beta_3=2$, $b_3=1+4b_4$, $b_4=0,\ 1$; where $C=\begin{bmatrix} Q^2P^x, & Q \\ P, & Q \end{bmatrix}$, with $x=1+2(2^{m-6}-1)$, -1, $1+2(\pm 2^{m-7}-1)$, respectively.

(C_3) $m=7$. From (10), $\beta_2=-\lambda-\kappa+2\beta_3$.

(a) $b_1=2b_2$. Here (a)$\sim$the four types, $\beta_3=0$; $a=1,\ 3$, $\kappa=b_2=\lambda=0$; $\kappa=\lambda=1$, $a=1$, $b_2=1$, $a=3$, $b_2=0$, with $C=\begin{bmatrix} P, & QP^{2x_1'} \\ P, & Q \end{bmatrix}$, where x_1' satisfies $\beta_3+\beta_3' \equiv x_1' \pmod 2$ and $b_2-b_2'-x_1'(\lambda-\kappa+a+2\beta_3+2b_2) \equiv 0 \pmod 4$.

(b) $b_1=1+2b_2$. Here (b)$\sim$the fourteen types, $a=1,\ 3$, $\kappa=0$; $\lambda=0$, $\beta_3=0$, $b_2=1,\ 3$; $\beta_3=1$, $b_2=1$; $\lambda=1$, $\beta_3=0$, $b_3=0,\ 1$; $\kappa=1$, $\beta_3=0$; $b_2=0$; $a=3$, $\lambda=0$; $a=1$, $\lambda=1$; $b_2=1,\ 3$, $a=3$, $\lambda=1$; with $C=\begin{bmatrix} P^x, & QP^{2x_1'} \\ P, & Q \end{bmatrix}$, where $x=1+2x_1$ and the variables satisfy $\beta_3+\beta_3'+x_1'(\lambda-\kappa)-\lambda x_1 \equiv 0 \pmod 2$, and $(b_2-b_2')x+(\lambda-\kappa)(ax_1-x_1')-2\beta_3(x_1+x_1')-2x_1'(1+b_2) \equiv 0 \pmod 4$. Hence in ($C$) there are twenty-five types for $m>7$; eighteen for $m=7$. The defining relations of these types are the following:

$Q^{-4}PQ^4=P^{1+2^{m-4}\kappa}$, $P^{2^{m-3}}=1$.

$Q^{-1}PQ=Q^{2a}P^{\omega_1+2^{m-6}b_3}$, $Q^{-2}PQ^2=Q^4P^{1+2^{m-5}\beta_3}$, $Q^8=1$; $a=1,\ 3$; $\omega_1=1$, $m>7$, $b_3=\beta_3=\kappa=0,\ 1$; $m=7$, $b_3=\beta_3=\kappa=0$; $\omega_1=-1$, $b_3=4$, $\beta_3=0,\ 2$, $\kappa=0$; $m=7$, $b_3=\beta_3=\kappa=0$.

$Q^{-1}PQ=Q^{2a}P^{-1+2^{m-5}b_3}$, $Q^{-2}PQ^2=Q^4P^{1+2^{m-5}(\beta_3-\lambda_2)}$, $Q^8=P^{2^{m-4}\lambda_2}$; $m>7$; $\kappa=0$, $b_3=\beta_3=0$, $a=1,\ 3$, $\lambda_2=0,\ 1$; $\kappa=1$; $\beta_3=1$, $\lambda_2=0,\ 1$; $a=1$, $b_3=1$; $a=3$, $b_3=0$; $\beta_3=3$, $\lambda_2=1$, $a=1$, $b_3=0,\ 1$; $m=7$, $a=3$, $\kappa=\beta_3=1$, $\lambda=b_3=0$.

$Q^{-1}PQ=Q^{2a}P^{-1-4a_1+2^{m-6}b_3}$, $Q^{-2}PQ^2=Q^4P^{1-4-2^{m-5}\beta_3}$, $Q^8=P^8$; $\beta_3=\kappa=0$; $a=1,\ 3$, $b_3=0,\ 2$; $m>8$, $a=1$, $b_3=1$; $m=8$, $a=3$, $b_3=1$; $\beta_3=\kappa=1$; $m>7$, $a=3$, $b_3=1,\ 3$; $m=7$, $a=1$, $b_3=2,\ 3$; $a=3$, $b_3=0,\ 2,\ 6$.

---

*For $m=8$, $b_3=1$, $a=3$.

†For $m=8$, $b_3-b_3'-2ax_1'[1+2a_1+2\beta_3+2b_3+2\kappa(a_1+1)] \equiv 0 \pmod 8$.

‡For $m=8$, $\beta_3=0$, $b_3=1+4b_4$; $\beta_3=2$, $b_3=3+4b_4$.

PART 2. Q^4 IS IN $\{P\}$ AND Q^2 IS NOT IN $\{P\}$.

We have $Q^4=P^{4\lambda}$. Two cases arise. They will be treated in separate sections.

§1. $\{P, Q\}$ IS OF ORDER 2^{m-1}.

Here $\{P, Q\}=G_{m-1}$ is determined from the types in III, Introduction, by replacing m by $m-1$. The types of G_{m-1} are now designated by (A), (B)... Thus writing $Q^{-1}PQ=P^{\omega+2^{m-\delta}\beta_1}...(1)$, $Q^4=P^{2^{m-4}\lambda_1}$, we have

(A) and (B), $\omega=\pm1$, $\beta_1=0, 1, 2$, $\lambda_1=0$; (C), $\omega=-1$, $\beta_1=0$, $\lambda_1=1$; and

$$Q^{-2}PQ^2=Q^{1+2^{m-4}\kappa}\ ...(2);$$

(D), $Q^{-1}PQ=Q^2P^{-1+2^{m-4}\beta_2}...(1)$, $Q^4=1$; $\beta_2=0, 1$, $\kappa=0, 1$;

(E), $Q^{-1}PQ=Q^2P^{1-2^{m-\delta}\kappa}\ ...(1)$, $Q^4=1$, $\kappa=0, 1$;

(F), $Q^{-1}PQ=Q^2P^{-1}...(1)$, $Q^4=P^{2^{m-4}}$;

(G), $Q^{-1}PQ=Q^2P^{-1+2^{m-\delta}\beta_2}...(1)$, $Q^4=P^4$, $\beta_2=0, 1$, $\kappa=0$; $\beta_2=0$, $\kappa=1$.

Let R be an operator of G_m, not in G_{m-1}. Since R^2 is in G_{m-1}, $R^2=Q^\mu P^\nu...(3)$. Then $G_m=[R, G_{m-1}]$. In G_m, $R^{-1}PR=Q^\rho P^\sigma...(4)$, and $R^{-1}QR=Q^\tau P^\omega...(5)$. Consider R with each G_{m-1}. For (A), (B), and (C), $[0, -y, x, 0, y]=[0, 0, x(\omega+2^{m-\delta}\beta_1)^y]...(6)$. Hence $[0, y, x]^s=[0, sy, (s-\theta_s)x\{\frac{1+(\omega)^y}{2}+(\omega)^{y-1}\times2^{m-\delta}\beta_1 y\}+\theta_s x\{1+2^{m-\delta}\beta_1(s-1)y\}...(7)$.

(A) From (3), (4), and (5), by means of (7), $b=1+2b_1$, $d=2^{m-\delta}d_1$, $\mu=2\mu_1$, $\nu=2\nu_1$. $(RP)^4$ is in $\{P\}$. Hence $a=2a_1$. Transforming (1), (3), (4), and (5) by R, we get $c=1+2c_1$, and the congruences

$$\beta_1 c_1\equiv0 \pmod 2...(8),\quad 2^{m-6}(d_1\mu_1+a_1\nu_1\beta_1)+b_1\nu_1\equiv0 \pmod{2^{m-\delta}}...(9),$$

$$2^{m-6}a_1d_1+b_1(1+b_1)+2^{m-6}(a_1b_1\beta_1-\beta_1\mu_1)\equiv0 \pmod{2^{m-\delta}}...(10),$$

and $d_1(1+b_1+c_1)\equiv\beta_1\nu_1 \pmod 2...(11)$.

$$[z,\ y,\ x]^{2s_1}=[0,\ 2\{y+z(\mu_1+c_1y+a_1x)\}s_1,\ 2s_1\{x+2^{m-6}\beta_1xy+z(\nu_1+b_1x+2^{m-\delta}b_1\beta_1xy+2^{m-6}d_1y+2^{m-6}a_1\beta_1(x-\theta_x))\}]...(12),\text{ and}$$

$$[z,\ y,\ x]^{2s_1+1}=[z,\ y(1+2s_1)+2s_1z(\mu_1+c_1y+a_1x),\ x(1+2s_1)+2^{m-4}\beta_1s_1x\{y+z(\mu_1+c_1y+a_1x)\}+2s_1\{2^{m-6}\beta_1xy+z[\nu_1+b_1x+2^{m-\delta}b_1\beta_1xy+2^{m-6}d_1y+2^{m-6}a_1\beta_1(x-\theta_x)]\}]...(13).$$

By means of the general correspondence $\begin{bmatrix} P', & Q', & R' \\ P, & Q, & R \end{bmatrix}$, where $P'=R^zQ^yP^x$, $Q'=R^{z'}Q^{y'}P^{x'}$, $R'=R^{z''}Q^{y''}P^{x''}$, the groups in (A) are simply isomorphic with the types given below.

(A_1) $b_1=2^{m-6}b_2$ $(b_2=0...3)$. x is odd, $x'=2^{m-6}x_1'$, $x''=2^{m-6}x_1''$, and the variables satisfy the congruences $y'+z'(\mu_1+c_1y')=k$ (k odd), $x_1'+d_1y'z'\equiv0$

(mod 2), $x_1''+d_1y''z''\equiv 0 \pmod 2$, $a_1z'+c_1(yz'-y'z)\equiv 0 \pmod 2$, $b_2(xz'-2^{m-6}x_1'z)+\beta_1(xy'-2^{m-6}x_1'y)+2a_1\beta_1(x_1z'-2^{m-7}x_1'z)+d_1(yz'-y'z)\equiv\beta_1'\{x+2^{m-6}(\beta_1xy+b_2xz+d_1yz)\} \pmod 4$, $a_1z''+c_1(yz''-y''z)\equiv a_1'[y'+z'(\mu_1+c_1y')] \pmod 2$, $b_2(xz''-2^{m-6}x_1''z)+\beta_1(xy''-2^{m-6}x_1''y)+2a_1\beta_1[2x_1z''-2^{m-6}x_1''z]+d_1(yz''-y''z)\equiv 2d_1a_1'z[y'+z'(\mu_1+c_1y')]+a_1'[2^{m-6}\beta_1x_1'y'+z'(d_1y'+2^{m-6}a_1\beta_1x_1')]+a_1'x_1'\times(1+2^{m-6}b_2z')+b_2'[x(1+2^{m-6}\beta_1y)+2^{m-6}z(b_2x+d_1y)] \pmod 4$, $c_1(y'z''-y''z')\equiv c_1'[y'+z'(\mu_1+c_1y')] \pmod 2$, $2^{m-6}\{b_2(x_1'z''-x_1''z')+\beta_1[x_1'y''-x_1''y'+a_1(x_1'z''-x_1''z')]\}+d_1(y'z''-y''z')\equiv c_1'\{x_1'[1+2^{m-6}(b_2z'+\beta_1y'+a_1\beta_1z')]+d_1y'z'\}+d_1'[x+2^{m-6}(\beta_1xy+b_2xz+d_1yz)] \pmod 4$, $y''+z''(\mu_1+c_1y'')\equiv\mu_1'[y'+z'(\mu_1+c_1y')] \pmod 2$, $x_1''+2^{m-6}\beta_1x_1''y''+z''[d_1y''+2^{m-6}x_1''(b_2+a_1\beta_1)]\equiv\mu_1'\{x_1'+2^{m-6}\beta_1x_1'y'+z'[d_1y'+2^{m-6}x_1'(b_2+a_1\beta_1)]\} \pmod 4$.

(A_2) $b_1=-1+2^{m-6}b_2$, ($b_2=0\ldots3$). In this case x and y' are odd, $z=z'=0$, $z''=1$, $x'=2^{m-5}x_2'$, and the variables satisfy the congruences $b_2x''+\beta_1x''y''+d_1y''\equiv 0 \pmod 2$, $\beta_1y'\equiv\beta_1'(1+2^{m-6}\beta_1y) \pmod 4$, $a_1'\equiv a_1+y(1+c_1) \pmod 2$, $b_2x-\beta_1\{xy''+x''y-2a_1(x_1+x'')-2c_1x''y+xy\}+d_1y\equiv 2a_1'x_2'+b_2'x(1+2^{m-6}\beta_1y) \pmod 4$, $c_1\equiv c_1' \pmod 2$, $2x_2'(1+c_1)-\beta_1x''(y'-2c_1)+d_1y'\equiv d_1'x(1+2^{m-6}\beta_1y) \pmod 4$, $\mu_1+y''(1+c_1)+a_1x''\equiv\mu_1' \pmod 2$, $b_2x''+d_1y''+\beta_1[a_1(x''-\theta_{x''})-x''y'']+2\nu_3\equiv 2\mu_1'x_2'+\nu_2'x(1+2^{m-6}\beta_1y) \pmod 4$. There are eighteen types in (A_1), twenty-eight in (A_2). These are tabulated below.

$Q^{-1}PQ=P^{1+2^{m-5}\beta_1}$, $R^{-1}PR=Q^{2a_1}P^{\omega_1+2^{m-5}b_2}$, $R^{-1}QR=Q^{1+2c_1}P^{2^{m-5}d_1}$,

$R^2=Q^{2\mu_1}P^{2^{m-4}\nu_2}$, $Q^4=1$, $P^{2^{m-3}}=1$.

a_1	c_1	μ_1	ν_2	b_2	d_1	β_1	ω_1	a_1	c_1	μ_1	ν_2	b_2	d_1	β_1	a_1	c_1	μ_1	ν_2	b_2	d_1	β_1
0,1	0	0	0	0	0	0,1,2	1	0	1	0	0	2	2	2	0	1	1	0	0	0	0,1
0	0,1	0	0	2	0	0	1	0	0	0	1	0	0	0,2	0,1	1	1	0	2	0	0
0	1	0	0	0	0,1	0,2	1	1	1	0	1	0	0	0,2	0	1	0	0	0	2	0
0	1	1	0	0,2	0	0	1	0	1	0	1	0	0	0	0	1	0	0	2	0	0
0,1	0	0	0	0	2	0,1	1	0,1	1	0	0	0	0	0,2	0	0	0	0	2	1	1
0	0	0	0	0	0,1	0,2	−1	1	1	0	0	0	2	0,2	0	1	1	0	0	2	0
0	0	0	0	2	0	0,2	−1	0	0	0	0	0	0	1	1	1	1	0	2	2	2
							$\omega_1=-1$														
																	$\omega_1=-1$				

(B) From (3) and (4), by means of (7), $a=2a_1$, $b=1+2b_1$, $\mu=2\mu_1$, $\nu=2\nu_1$. Also $(RQ)^4$ is in $\{P\}$; hence $c=1+2c_1$, and $d=2d_1$. Transforming (1), (3), (4), and (5) by R, we obtain

$\beta_1(a_1+c_1)\equiv 0 \pmod 2$...(8),

$\nu_1b_1+2^{m-6}\beta_1(\mu_1d_1+a_1\nu_1)\equiv 0 \pmod{2^{m-5}}$...(9),

$d_1(1+b_1)-\nu_1(1-2^{m-6}\beta_1)\equiv 0 \pmod{2^{m-5}}$...(10), and

$b_1(1+b_1)+2^{m-6}\beta_1[a_1(b_1+d_1)+\mu_1]\equiv 0 \pmod{2^{m-5}}$...(11).

$[z, y, x]^{2s_1}=[0,\ 2s_1\{y+z(\mu_1+c_1y+a_1x)\},\ 2s_1\{(\phi_y+(-1)^{y-1}2^{m-6}\beta_1y)x+z[\nu_1+(-1)^y(b_1x+d_1\theta_y)(1-2^{m-5}\beta_1y)+2^{m-6}\beta_1(a_1(x-\theta_x)+d_1(y-\theta_y))]\}]$. (12),

and $[z, y, x]^{2s_1+1}=[z,\ y(1+2s_1)+2s_1z(\mu_1+c_1y+a_1x),\ x\{1-2^{m-4}\beta_1\ s_1[y+z(\mu_1+c_1y+a_1x))\}+2s_1\{x(\phi_y+(-1)^{y-1}2^{m-6}\beta_1y)+z[\nu_1+(-1)^y(b_1x+d_1\theta_y)(1-2^{m-5}\beta_1y)+2^{m-6}\beta_1(a_1(x-\theta_x)+d_1(y-\theta_y))]\}]$...(13).

The groups in (B) for ν_1 other than 0 are isomorphic with those for $\nu_1=0$; with $C=\begin{bmatrix}P, & Q, & R'\\ P, & Q, & R\end{bmatrix}$, where $z''=1$, y'' is even for b_1 even, and odd for b_1 odd, and x'' and y'' satisfy the congruence $\nu_1+x''\theta_{y''}+(-1)^{y''}(b_1x''+d_1\theta_{y''})(1-2^{m-5}\beta_1y'')+2^{m-6}\beta_1[a_1(x''-\theta_{x''})+d_1(y''-\theta_{y''})-(-1)^{y''}x''y'']\equiv\nu_1'\pmod{2^{m-4}}$.

(B_1) $b_1=2^{m-6}b_2$. From (10), $d_1=2^{m-5}d_3$.

(B_2) $b_1=-1+2^{m-6}b_2$. All the groups in (B_2) are isomorphic with those where $d_1=0, 1, 2$; with $C=\begin{bmatrix}P, & QP^{x'}, & RP^{x''}\\ P, & Q, & R\end{bmatrix}$, where x' and x'' satisfy $\beta_1x'\equiv0\pmod 2$, $\mu_1+a_1x''\equiv\mu_1'\pmod 2$, $b_2x''+\beta_1a_1(x''-\theta_{x''})\equiv\beta_1\mu_1'x'\pmod 4$, and $x''+b_1x'+d_1-2^{m-6}\beta_1[x''(1+2c_1+2a_1x')+a_1(x'-\theta_{x'})+3c_1'x']\equiv d_1'\pmod{2^{m-4}}$.

The groups in (B) are simply isomorphic with the types given below, or with (A). The correspondence is given by $C=\begin{bmatrix}P', & Q', & R'\\ P, & Q, & R\end{bmatrix}$, where x and y' are odd. For (B_1), y is even. For (B_2), $y+z\equiv0\pmod 2$; except for the groups$\sim(A)$, where for (B_1) x and y'' are odd, y and y' even, $z'=1, x'=2^{m-5}x_1'$; and for (B_2) $P'=P$, $Q'=RQP^{x'}$, $R'=R$. The variables satisfy the congruences derived in the usual way from the isomorphism. These congruences are somewhat complicated, but similar to those derived in previous cases. They are omitted for the sake of brevity.

There are in (B_1) twenty-two types; in (B_2) two for $m>7$, one for $m=7$, viz:

(B_1) $Q^{-1}PQ=P^{-1+2^{m-5}\beta_1}$, $R^{-1}PR=Q^{2a_1}P^{1+2^{m-5}b_2}$, $R^{-1}QR=Q^{1+2c_1}P^{2^{m-4}d_1}$, $R^2=Q^{2\mu_1}$, $P^{2^{m-3}}=1$.

a_1	c_1	μ_1	d_3	b_2	β_1	a_1	c_1	μ_1	d_3	b_2	β_1	a_1	c_1	ν_1	d_3	b_2	β_1
0	0	0	0	0	0,1,2	0	0	0	0	2	0	0	0	0	1	0	0,1
0	1	0	0	0,2	0,2	0	1	0	1	0	0	1	0	0	0	0,2	0,2
1	0	0	1	0,2	0	1	1	0	0,1	0	1	1	0	1	1	0,2	0

$b_2=2$, $\beta_1=2$.

(B_2) $Q^{-1}PQ=P^{-1+2^{m-5}}$, $R^{-1}PR=Q^2P^{-1+2^{m-4}b_2}$, $R^{-1}QR=Q^3P^2$, $R^2=Q^4$, $P^{2^{m-3}}=1$; $m>7$, $b_2=0, 1$; $m=7$, $b_2=0$.

(C) From (3) and (4) by means of (7), $a=2a_1$, $b=1+2b_1$, $\mu=2\mu_1$, $\nu=2\nu_1$. The transformation of (1) by R shows that $c=1+2c_1$, $a_1=0$. Also RQ must be of order 2^{m-3} at most; hence $d=2d_1$. Transforming (3), (4), and (5) by R, we obtain

$b_1\nu_1+2^{m-6}c_1\mu_1\equiv 0 \pmod{2^{m-5}}$...(8),

$b_1(1+b_1)\equiv 0 \pmod{2^{m-5}}$...(9),

$d_1(1+b_1)\equiv\nu_1 \pmod{2^{m-5}}$...(10),

$[z, y, x]^{2s_1}=[0, 2s_1\{y+z(\mu_1+c_1y)\}, 2s_1\{\phi_y x+(-1)^y z(d_1\theta_y+b_1x)+\nu_1 z\}]$...(11),

and $[z, y, x]^{2s_1+1}=[z, y(1+2s_1)+2s_1z(\mu_1+c_1y), x+2s_1\{\phi_y x+(-1)^y z(d_1\theta_y+b_1x)+\nu_1 z\}]$...(12).

If $Q'^4=(RQ^{y'}P^{x'})^4=1$, and $Q'^2\neq\{P\}$, then $\mu_1+y'(1+c_1)$ is odd (13), and $x'\phi_{y'}+(-1)^{y'}(d_1\theta_{y'}+b_1x')+2^{m-6}[\mu_1+y'(1+c_1)]\equiv 0 \pmod{2^{m-5}}$. The groups satisfying these conditions do not belong in (C).

The groups for ν_1 other than 0 are isomorphic with those for $\nu_1=0$, through $C=\begin{bmatrix} P, & Q, & R' \\ P, & Q, & R \end{bmatrix}$, where $z''=1$, y'' is even for b_1 even, and odd for b_1 odd, and x'' and y'' satisfy the congruence $2^{m-5}[\mu_1+\mu_1'+y''(1+c_1)]+x''\phi_{y''}+(-1)^{y''}[d_1\theta_{y''}+b_1x'']\equiv\nu_1'-\nu_1 \pmod{2^{m-4}}$.

(C_1) $b_1=2^{m-5}b_2$. From (10), $d_1=2^{m-5}d_2$. $(C_1)\sim$the four types given below, with $C=\begin{bmatrix} P', & Q', & R' \\ P, & Q, & R \end{bmatrix}$, where x and y' are odd, y and y'' even, $z'=z=0$, $z''=1$, $x''=2^{m-5}x''$, and the variables satisfy $y_1\equiv 0 \pmod 2$, $d_2+d_2'+b_2x'+x_1''\equiv 0 \pmod 2$, $y_1''(1+c_1)-\mu_1y_1'\equiv x_1'' \pmod 2$.

(C_2) $b_1=-1+2^{m-5}b_2$. All the groups in (C_2) are isomorphic with those where $d_1=0$, by $C=\begin{bmatrix} P, & QP^{x'}, & RP^{2x_1''} \\ P, & Q, & R, \end{bmatrix}$, where x' and x_1'' satisfy $d_1'\equiv d_1+b_1x'+2x_1'' \pmod{2^{m-4}}$. Also $(C_2)\sim(C_1)$, with $C=\begin{bmatrix} P, & Q, & RQP^{2^{m-5}} \\ P, & Q, & R \end{bmatrix}$.

In (C) there are the following four types:

$$Q^{-1}PQ=P^{-1},\ R^{-1}PR=P^{1+2^{m-4}b_2},\ R^{-1}QR=Q^{1+2c_1},\ R^2=1,\ Q^4=P^{2^{m-4}},\ P^{2^{m-3}}=1,$$
$$c_1=0, 1,\ b_2=0, 1.$$

For (D), (E), (F), and (G),

$[0, -2y_1, x, 0, 2y_1]=[0, 0, x(1+2^{m-4}\kappa y_1)]$...(6), and

$[0, 2y_1, x]^s=[0, 2sy_1, sx+2^{m-5}\kappa xy_1(s-\theta_s)]$...(7).

(D) From (1), $[0, -y, x, 0, y]=[0, 2xy, (-1)^y x+2^{m-5}\kappa\{x(y-\theta_y)+y(x-\theta_x)\}+2^{m-4}\beta_2 xy]$...(8). Hence

$[0, y, x]^s=[0, sy+(s-\theta_s)xy, \theta_s x+(s-\theta_s)\{x\phi_y+2^{m-6}\kappa[x(y-\theta_y)+y(x-\alpha_x)]+2^{m-5}\beta_2 xy\}]$...(9).

From (3) and (4) by means of (9), $\mu+\nu\equiv 0 \pmod 2$, $a=2a_1$, $b=1+2b_1$. If $c=2c_1$, or if c and d were both odd, Q^2, transformed by R^2 would result in $P^{2N}=Q^2P^{2^{m-4}M}$, which is impossible. Hence $c=1+2c_1$, and $d=2d_1$. Transforming (1), (3), (4), and (5) by R, we get $\mu=2\mu_1$, $\nu=2\nu_1$, and the congruences

$\kappa(a_1+b_1+c_1+d_1)\equiv 0 \pmod 2$...(10),

$\nu_1 b_1+2^{m-6}\kappa(\mu_1 d_1+\nu_1 a_1)\equiv 0 \pmod{2^{m-5}}$...(11),

$b_1(1+b_1)+2^{m-6}\kappa[a_1(b_1+d_1)+\mu_1]\equiv 0 \pmod{2^{m-5}}$...(12),

$d_1(1+b_1)-\nu_1+2^{m-6}\kappa[d_1(a_1+c_1)+\nu_1]\equiv 0 \pmod{2^{m-3}}$...(13),

$[z,\ y,\ x]^{2s_1}=[0,\ 2s_1\{y(1+x)+z(\mu_1+c_1y+a_1x)\},\ 2s_1\{x\phi_y+2^{m-6}\kappa[x(y-\theta_y)+y(x-\theta_x)]+2^{m-5}\beta_2xy+z[\nu_1+(b_1x+d_1\theta_y)((-1)^y+2^{m-5}\kappa y)+2^{m-6}\kappa(a_1(x-\theta_x)+d_1(y-\theta_y))]\}]$...(14), and

$[z,\ y,\ x]^{2s_1+1}=[z,\ y(1+2s_1)+2s_1\{xy+z(\mu_1+c_1y+a_1x)\},\ x\{1+2^{m-4}\kappa s_1z(\mu_1+c_1y+a_1x)\}+2s_1\{x\phi_y+2^{m-6}\kappa[x(y-\theta_y)+y(x-\theta_x)]+2^{m-5}\beta_2xy+z[\nu_1+(b_1x+d_1\theta_y)((-1)^y+2^{m-5}\kappa y)+2^{m-6}\kappa(a_1(x-\theta_x)+d_1(y-\theta_y))]\}]$...(15).

The groups for ν_1 other than 0 are isomorphic with those for $\nu_1=0$ and $C=\begin{bmatrix} P, & Q, & R' \\ P, & Q, & R \end{bmatrix}$, where $z''=1$, y'' is even for b_1 even, odd for b_1 odd, and x'' and y'' satisfy the congruence $\nu_1+x''\phi_{y''}+(-1)^{y''}(b_1x''+d_1\theta_{y''})+2^{m-6}\{\kappa[(x''+d_1)(y''-\theta_{y''})+(y''+a_1)(x''-\theta_{x''})+2y''(b_1x''+d_1\theta_{y''})]+2\beta_2x''y''\}\equiv\nu_1' \pmod{2^{m-4}}$. For $\mu_1=1$, the groups where $a_1=0$ for b_1 even, and $a_1=1$ for b_1 odd, are isomorphic to (A), (B), or (C), with $C=\begin{bmatrix} P, & R, & Q^{y''}P^{x''} \\ P, & Q, & R \end{bmatrix}$, where y'' and x'' are odd and satisfy $\kappa(y_1''+x_1'')+\beta_2\equiv 0 \pmod 2$. Also the groups for $a_1=0$, b_1 odd are isomorphic to (A), (B), or (C), with $C=\begin{bmatrix} P, & R, & QP \\ P, & Q, & R \end{bmatrix}$.

(D_1) $b_1=2^{m-6}b_2$. From (13), $d_1=2^{m-5}d_2$.

(D_2) $b_1=-1+2^{m-5}b_2$. All the groups in $(D_2)\sim$those where $d_1=0, 1$, with $C=\begin{bmatrix} P, & QP^{2x_1'}, & RP^{x''} \\ P, & Q, & R \end{bmatrix}$, where x_1' and x'' satisfy $\mu_1+a_1x''\equiv\mu_1' \pmod 2$, $b_2x''+2a_1\kappa x_1''\equiv 2\kappa\mu_1'x_1' \pmod 4$, $x''-2x_1'+d_1-d_1'+2^{m-5}\{x''(\beta_2+\kappa c_1)+x_1'(b_2+\kappa+\kappa c_1')\}+2^{m-6}\kappa(x''+\theta_{x''})\equiv 0 \pmod{2^{m-4}}$, $c_1-x''\equiv c_1' \pmod 2$.

The groups in (D) are simply isomorphic with the types given below, or with those in the preceding cases through $C=\begin{bmatrix} P', & Q', & R' \\ P, & Q, & R \end{bmatrix}$, x and y' are odd, x' even, $z'=0$, $z''=1$. For (D_1), y and y'' are even, and $x''=2^{m-5}x_1''$, and for (D_2), y'' is odd; except for the groups$\sim(A)$, (B), or (C), where for (D_1) x is odd and y even, and for (D_2), $P'=P$, $Q'=RQP^{x'}$, $R'=R$. It has been verified that the variables, limited as above, satisfy the congruences derived in the usual way from C by means of the transforming relations of the group.

There are seventeen types in (D_1), and five in (D_2). They follow:

$Q^{-1}PQ=Q^2P^{-1+2^{m-4}\beta_2}$, $Q^{-2}PQ^2=P^{1+2^{m-4}\kappa}$, $R^{-1}PR=Q^{2a_1}P^{\omega_1+2^{m-4}b_2}$,

$R^{-1}QR=Q^{1+2c_2}P^{2^{m-4}d_1}$, $R^2=Q^{2\mu_1}$, $Q^4=1$, $P^{2^{m-3}}=1$.

a_1	c_1	μ_1	d_2	b_2	κ	β_2	ω_1
0	0	0	0	0	0	0,1	1
1	1	0	0	0	0	0,1	
0	0	0	1	1	0	0,1	
1	1	0	1	1	0	0,1	
0	0	0	0	1	0	0	1
1	1	0	0	1	0	0	
0	0	0	1	0	0	0	
1	1	0	1	0	0	0	
0	0	0	0	0	0	0,1	1
1	1	0	0	0	0	0,1	
0	0	1	1	0	0	0	
0	0	0	0	0	0	0	−1
0	0	0	0	1	0	0,1	−1

$Q^{-1}PQ=Q^2P^{-1+2^{m-4}\beta_2}$, $Q^{-2}PQ^2=P$, $R^{-1}PR=P^{-1+2^{m-4}}$, $R^{-1}QR=Q^3P^2$,

$R^2=1$, $Q^4=1$, $P^{2^{m-3}}=1$, $\beta_2=0, 1$.

(E) From (1), $[0, -y, x, 0, y]=[0, 2xy, x+2^{m-5}\kappa\{x(y-\theta_y)-\theta_{xy}\}]$...(8), and $[0, y, x]^s=[0, sy+(s-\theta_s)xy, sx+2^{m-6}(s-\theta_s)\kappa\{x(y-\theta_y)-\theta_{xy}\}]$...(9). From (3), (4), and (5), $\mu=2\mu_1$, $\nu=2\nu_1$, $b=1+2b_1$, $d=2^{m-5}d_1$. The operation $(RP)^4$ is in $\{P\}$, so $a=2a_1$. Transformation of (1), (3), (4), and (5) by R gives $c=1+2c_1$, and the congruences

$d_1+\kappa(b_1+c_1)\equiv 0 \pmod 2$...(10),

$b_1\nu_1+2^{m-6}(d_1\mu_1+\kappa\nu_1a_1)\equiv 0 \pmod{2^{m-5}}$...(11),

$b_1(1+b_1)+2^{m-6}[a_1(d_1+\kappa b_1)+\kappa\mu_1]=0 \pmod{2^{m-5}}$...(12), and

$d_1(1+b_1+c_1)\equiv 0 \pmod 2$...(13). Also,

$[z, y, x]^{2s_1}=[0,\ 2s_1\{y(1+x)+z(\mu_1+c_1y+a_1x)\},\ 2s_1\{x+2^{m-6}\kappa[x(y-\theta_y)-\theta_{yx}]+z[\nu_1+b_1x+2^{m-6}\kappa a_1(x-\theta_x)+2^{m-6}d_1y]\}]$...(14),

$[z, y, x]^{2s_1+1}=[z,\ y(1+2s_1)+2s_1\{xy+z(\mu_1+c_1y+a_1x)\},\ x(1+2s_1)+2s_1\{2^{m-6}\kappa\times[x(y-\theta_y)-\theta_{yx}]+z[\nu_1+b_1x+2^{m-6}\kappa(a_1(x-\theta_x)+2(\mu_1+c_1y+a_1x))+2^{m-6}d_1y]\}]$...(15).

(E_1) $b_1=2^{m-6}b_2$. (E_2) $b_2=-1+2^{m-6}b_2$. From (11), $\nu_1=2^{m-6}\nu_2$. The groups in (E_1) and for b_2 odd, in (E_2), for ν_1 other than 0 are isomorphic with those for $\nu_1=0$. When b_2 is even in (E_2), and $\nu_2=3$, the groups are isomorphic with those for $\nu_2=1$. $C=\begin{bmatrix}P, & Q, & R'\\ P, & Q, & R\end{bmatrix}$, where $z''=1$, and x'' and y'' satisfy $\nu_1+x''(1+b_1)+2^{m-6}\kappa[a_1(x''-\theta_{x''})+x''(y''-\theta_{y''})-\theta_{x''y''}]+2^{m-6}d_1y''\equiv\nu_1'$ (mod 2^{m-4}). For $a_1=0$, $\mu_1=1$, $\nu_1=0, 2$, (E)$\sim$(A), (B) or (C); and for $\nu_1=0$, $C_1=\begin{bmatrix}P, & R, & Q\\ P, & Q, & R\end{bmatrix}$; and for $\nu_1=2$, $C=\begin{bmatrix}RP, & Q, & R\\ P & Q, & R\end{bmatrix}$. For $a_1=1$, and $\mu_1+c_1\equiv 0 \pmod 2$, ($E$)$\sim$($A$), ($B$) or ($C$) with $C=\begin{bmatrix}P, & RQP^{x'}, & R\\ P, & Q, & R\end{bmatrix}$, where x' satisfies $x'(1+b_1)+2^{m-6}(\kappa x'+d_1)\equiv 0 \pmod{2^{m-5}}$. For $a_1=0$, 1, $\mu_1=0$, $c_1=1$, (E_1), and for

$\kappa=0$, $(E_2)\sim(A)$, (B), or (C). For (E_1), $C=\begin{bmatrix} RP, & Q, & R \\ P, & Q, & R \end{bmatrix}$; for (E_2), $a_1=0$, $C=\begin{bmatrix} RP, & QP^{2^{m-5}x_1'}, & R \\ P, & Q, & R \end{bmatrix}$, where $x_1' \equiv d_2 \pmod 2$; for (E_2), $a_1=1$, $C=\begin{bmatrix} QP, & RP, & R \\ P, & Q, & R \end{bmatrix}$. For $a_1=0$, $\mu_1+c_1\equiv 0 \pmod 2$, $\kappa=0$, $(E_2)\sim(D)$ with $C=\begin{bmatrix} P, & RQ, & R \\ P, & Q, & R \end{bmatrix}$. For $\kappa=d_1=0$ in (E_1) and for $\kappa=1$ in (E), for $a_1=\mu_1=c_1=0$, the groups where $b_2=2\sim$those where $b_2=0$; with $C=\begin{bmatrix} QP, & Q, & R \\ P, & Q, & R \end{bmatrix}$, for (E_1), $\kappa=0$, and for (E_2), $\kappa=1$; and with $C=\begin{bmatrix} P^3, & Q, & RQ^2 \\ P, & Q, & R \end{bmatrix}$, for (E_1), $\kappa=1$. In (E_2), for $\kappa=1$, the groups where $a_1=0, 1$, $\mu_1=0$, $c_1=1\sim$those where $a_1=c_1=\mu_1=0$, by $C=\begin{bmatrix} P, & Q, & RQ^{y''}P \\ P, & Q, & R \end{bmatrix}$, where for $a_1=0$, $y''\equiv 2b_3' \pmod 4$, and for $a_1=1$, $y''=1+2y_1''$, and y'' satisfies $b_3+d_2+y_1''\equiv 0 \pmod 2$, and $b_3+b_3'+y_1''\equiv 0 \pmod 2$, ($b_2=1+2b_3$, $b_2'=2b_3'$). Hence there are five types in (E_1), and two in (E_2), viz:

$$Q^{-1}PQ=Q^2P^{1-2^{m-5}\kappa}, \quad Q^{-2}PQ^2=P^{1+2^{m-4}\kappa}, \quad R^{-1}PR=P^{\omega_1+2^{m-4}b_2}, \quad R^{-1}QR=QP^{2^{m-5}d_1}, \quad R^2=1, \quad Q^4=1, \quad P^{2^{m-3}}=1.$$

$\omega_1=1$; $\kappa=0, 1$, $d_1=0, 1$, $b_2=0$; $\kappa=0$, $d_1=0$, $b_2=1$; $\omega_1=-1$, $\kappa=1$, $b_2=0$, $d_1=1, 3$.

(F) From (1), $[0, -y, x, 0, y]=[1, 2x\theta_y, (-1)^y x-2^{m-5}\kappa(xy-\theta_{xy})]$...(8), and $[0, y, x]^s=[0, sy+(s-\theta_s)x\theta_y, \theta_s x+(s-\theta_s)\{x\phi_y-2^{m-6}\kappa(xy-\theta_{xy})\}]$...(9).

From (3) and (4), by means of (9), $\mu+\nu\equiv 0 \pmod 2$, $a=2a_1$, $b=1+2b_1$. Transformation of (1), (3), (4), and (5) by R gives $c=1+2c_1$, $d=2d_1$, $\mu=2\mu_1$, $\nu=2\nu_1$, and the congruences

$(1+\kappa)(a_1+b_1+c_1+d_1)\equiv 0 \pmod 2$...(10),

$b_1\nu_1+2^{m-6}\{\mu_1 c_1+(d_1\mu_1+a_1\nu_1)(1+\kappa)\}\equiv 0 \pmod{2^{m-5}}$...(11),

$b_1(1+b_1)+2^{m-6}\{a_1(1+b_1+c_1+d_1)+\kappa(d_1+\mu_1+a_1b_1)\}\equiv 0 \pmod{2^{m-5}}$...(12),

$d_1(1+b_1)-\nu_1+2^{m-6}\{\nu_1+d_1(a_1+c_1)\}(1+\kappa)\equiv 0 \pmod{2^{m-5}}$...(13),

$[z, y, x]^{2s_1}=[0, 2s_1\{y+x\theta_y+z(\mu_1+c_1y+a_1x)\}, 2s_1\{x\phi_y-2^{m-6}\kappa(xy-\theta_{xy})+z[\nu_1+(-1)^y(b_1x+d_1\theta_y)(1-2^{m-5}\kappa y)+2^{m-6}c_1\kappa(x-\theta_x)+2^{m-6}d_1(1+\kappa)(y-\theta_y)]\}]$ ...(14), and

$[z, y, x]^{2s_1+1}=[z, y(1+2s_1)+2s_1\{x\theta_y+z(\mu_1+c_1y+a_1x)\}, x\{1+2^{m-4}\kappa s_1[y+x\theta_y+z(\mu_1+c_1y+a_1x)]\}+2s_1\{x\phi_y-2^{m-6}\kappa(xy-\theta_{xy})+z[\nu_1+(-1)^y(b_1x+d_1\theta_y)(1-2^{m-5}\kappa y)+2^{m-6}a_1\kappa(x-\theta_x)+2^{m-6}d_1(1+\kappa)(y-\theta_y)]\}]$...(15).

If $Q'^4=(RQ^{y'}P^{x'})^4=1$, and $\{Q'^2\}\neq\{P^4\}$, then

$\mu_1+y'(1+c_1)+a_1x'+x'\theta_{y'}$ is odd...(16), and

$\nu_1+x'\phi_{y'}+(-1)^{y'}(b_1x'+d_1\theta_{y'})-2^{m-6}\{\kappa[y'(x'+2b_1x'+2d_1\theta_{y'})-\theta_{x'y'}]-\kappa a_1(x'-\theta_{x'})-d_1(1+\kappa)(y'-\theta_{y'})\}=2^{m-6}k$ (k odd)...(17).

The groups satisfying (16) and (17) do not belong in (F).

The groups for ν other than 0, are isomorphic with those for $\nu_1=1$, with $C=\begin{bmatrix}P, & Q, & R'\\ P, & Q, & R\end{bmatrix}$, where $z''=1$, y'' is even for b_1 even, and odd for b_1 odd, and x'' and y'' satisfy the congruence $2^{m-6}\mu_1'+\nu_1'\equiv 2^{m-6}[\mu_1+y''(1+c_1)+a_1x''+x''\theta_{y''}]+\nu_1+x''\phi_{y''}+(-1)^{y''}(b_1x''+d_1\theta_{y''})(1-2^{m-5}\kappa)-2^{m-6}\{\kappa[x''y''-\theta_{x''y''}+a_1(x''-\theta_{x''})]+d_1(1+\kappa)(y''-\theta_{y''})\}$ (mod 2^{m-4}). For $\mu_1=1$, $a_1=0$, and for $a_1=1$, $\mu_1+c_1\equiv 0$ (mod 2), the groups are isomorphic with (A), (B), or (C), and for $a_1=0$, $C=\begin{bmatrix}P, & R, & Q\\ P, & Q, & R\end{bmatrix}$, and for $a_1=1$, $C=\begin{bmatrix}P, & RQP^{x'}, & R\\ P, & Q, & R\end{bmatrix}$, where x' satisfies $b_1x'+d_1\equiv 0$ (mod 2^{m-5}).

(F_1) $b_1=2^{m-6}b_2$. From (13), $d_1=2^{m-5}d_2$.

(F_2) $b_1=-1+2^{m-6}b_2$. All the groups in (F_2)$\sim$those where $d_1=0, 1, 2$ with $C=\begin{bmatrix}P, & QP^{2x_1'}, & RP^{x''}\\ P, & Q, & R\end{bmatrix}$, where x_1' and x'' satisfy the congruences $2\mu_1x_1'(1+\kappa)-x''(a_1+b_2)+\kappa a_1(x''-\theta_{x''})\equiv 0$ (mod 4), $d_1'-d_1+2x_1'-x''+2^{m-6}\{c_1'[1+2x_1'(1+\kappa)]-c_1(1+2\kappa x'')+\kappa(x''+\theta_{x''})+2x_1'[a_1(1+\kappa)+b_2]+x''\}$ (mod 2^{m-4}), $c_1-x''\equiv c_1'$ (mod 2).

The groups in (F) are simply isomorphic with the types given below, or with those in preceding cases; where $Q^4=P^{2^{m-4}}$, and $C=\begin{bmatrix}P', & Q', & R'\\ P, & Q, & R\end{bmatrix}$. The variables x and y' are odd, x' and y even, $z'=0$, $z''=1$. For (F_1) y'' is even, and $x''=2^{m-5}x_1''$; for (F_2), $z=0$, and y'' is odd; except for the groups$\sim$(C), where x, y' are odd, x' even, $z=1$, $z'=0$, and for (F_1) y is even, for (F_2) y is odd. The variables specified have been proven to satisfy the congruence conditions derived from the relations of the group.

There are eight types in (F_1), and none in (F_2). The groups of (F_1) are given by

$Q^{-1}PQ=Q^2P^{-1}$, $Q^{-2}PQ^2=P^{1+2^{m-4}\kappa}$, $R^{-1}PR=Q^{2a_1}P^{1+2^{m-4}b_2}$, $R^{-1}QR=Q^{1+2c_1}P^{2^{m-4}d_2}$, $R^2=1$, $Q^4=P^{2^{m-4}}$, $P^{2^{m-3}}=1$.

$\kappa=0, 1$, $b_2=0, 1$, $a_1=c_1=d_2=0$; $\kappa=0$, $b_2=0, 1$, $d_2=0, 1$, $a_1=c_1=1$.

(G) From (1),

$[0, -y, x, 0, y]=[0, 2x\theta_y, (-1)^yx+2^{m-5}\{\kappa(xy-\theta_{xy})+\beta_2x\theta_x\}]$...(8), and

$[0, y, x]^s=[0, sy+(s-\theta_s)x\theta_y, \theta_sx+(s-\theta_s)\{x\phi_y+2^{m-6}[\kappa(xy-\theta_{xy})+\beta_2x\theta_y]\}]$...(9).

From (3) and (4), by means of (9), $\mu+\nu\equiv 0$ (mod 2), and either $a=2a_1$, $b=1+2b_1$, or $a=1+2a_1$, $b=2b_1$. For the latter, $R^{-1}PR=Q^{1+2a_1}P^{2b_1}$, from which Q may be obtained in terms of R and P. Therefore the group is generated by R and P alone, and equals $\{R, P\}$, which belongs in §2 as well as in §1.

(G_1) $a=2a_1$, $b=1+2b_1$. Transforming (1), (3), (4), and (5), and $Q^4=P^4$ by R, we get $c=1+2c_1$, $d=2d_1$, $\mu=2\mu_1$, $\nu=2\nu_1$, and the congruences

$a_1+b_1-c_1-d_1+2^{m-6}\beta_2(a_1+d_1)\equiv 0 \pmod{2^{m-5}}$...(10),

$\mu_1(c_1+d_1)+\nu_1(a_1+b_1)+2^{m-6}[\kappa(\mu_1 d_1+\nu_1 a_1)+\beta_2\mu_1 d_1]\equiv 0 \pmod{2^{m-5}}$...(11),

$b_1(1+b_1)+a_1(1+b_1+c_1+d_1)+2^{m-6}[\kappa(a_1b_1+a_1d_1+\mu_1)+\beta_2 a_1 d_1]\equiv 0 \pmod{2^{m-5}}$...(12),

$d_1(1+a_1+b_1+c_1)+c_1(1+c_1)+2^{m-6}[\kappa(a_1d_1+c_1d_1+\nu_1)+\beta_2(c_1d_1+\nu_1)]\equiv 0 \pmod{2^{m-5}}$...(13),

$a_1+b_1-c_1-d_1\equiv 0 \pmod{2^{m-6}}$...(14),

$[z, y, x]^{2s_1}=[0, 2s_1\{y+x\theta_y+z(\mu_1+c_1y+a_1x)\}, 2s_1\{x\phi_y+2^{m-6}[\kappa(xy-\theta_{xy})+\beta_2 x\theta_y]+z[\nu_1+(b_1x+d_1y)(1+2^{m-5}y(\kappa+\beta_2))+2^{m-6}(\kappa a_1(x-\theta_x)+d_1(\kappa+\beta_2)\times(y-\theta_y))]\}]$...(15), and

$[z, y, x]^{2s_1+1}=[z, y(1+2s_1)+2s_1\{x\theta_y+z(\mu_1+c_1y+a_1x)\}, x[1+2^{m-4}\kappa s_1 z(\mu_1+c_1y+a_1x)]+2s_1\{x\phi_y+2^{m-6}[\kappa(xy-\theta_{xy})+\beta_2 x\theta_y]+z[\nu_1+(b_2x+d_1y)(1+2^{m-5}y(\kappa+\beta_2))+2^{m-6}(\kappa a_1(x-\theta_x)+d_1(\kappa+\beta_2)(y-\theta_y))]\}]$...(16).

If $Q'^{2^{m-3}}=(RQ^{y'}P^{x'})^{2^{m-3}}\neq 1$, and $Q'^2\neq\{P\}$, then $\mu_1+y'(1+c_1)+x'(a_1+\theta y')=k$...(17), and $x'\phi_{y'}+b_1x'+d_1y'=k'$...(18) ($k$ and k' odd). The groups satisfying (17) and (18) do not belong to (G).

The groups for ν_1 other than $0\sim$those for $\nu_1=0$, with $C=\begin{bmatrix}P, & Q, & R'\\ P, & Q, & R\end{bmatrix}$ where $z''=1$, y'' is even for b_1 even, and odd for b_1 odd; and x'' and y'' satisfy the congruence $\nu_1'+\mu_1'\equiv\nu_1+\mu_1+y''(1+c_1+d_1)+x''(1+a_1+b_1)+2^{m-6}\{\kappa[a_1(x''-\theta_{x''})+x''y''-\theta_{x''y''}]+\beta_2x''\theta_{y''}+(\kappa+\beta_2)[d_1(y''-\theta_{y''}+2y''(b_1x''+d_1y'')]\}$ (mod 2^{m-4}).

(G_1) $b_1=2^{m-5}b_2$, $d_1=2^{m-5}d_2$, $a_1=c_1=\mu_1=0$. (G_1)$\sim$the five types given below and $C=\begin{bmatrix}Q, & P, & R\\ P, & Q, & R\end{bmatrix}$ and $\begin{bmatrix}P, & QP^{2^{m-6}x_1'}, & RQ^{2y_1''}P^{2x_1''}\\ P, & Q, & R\end{bmatrix}$, where x_1', x_1'' and y_1'' satisfy $x_1'+\beta_2'-\beta_2+2^{m-6}x_1'(\kappa+\beta_2)\equiv 0 \pmod 4$, $y_1''+x_1''\equiv 0 \pmod{2^{m-5}}$, $b_2'+b_2+\kappa y_1''\equiv 0 \pmod 2$, and $d_2'+d_2+(\kappa+\beta_2)x_1''\equiv 0 \pmod 2$.

(G_2) $b_1=-1+2^{m-6}b_2$, $d_1=-1+2^{m-6}(2d_3+b_2)$, $a_1=c_1=1$, $\mu_1=0$. (G_2)$\sim$(G_1) and $C=\begin{bmatrix}P, & Q, & RQ^{y''}P\\ P, & Q, & R\end{bmatrix}$, where $y''=-1+2^{m-6}y_1''$ and satisfies $y_1''+2d_3+2^{m-6}b_2y_1''+\beta_2(-1+2^{m-6}y_1'')\equiv 0 \pmod 4$. There are five types in (G). They are defined as follows:

$Q^{-1}PQ=Q^2P^{-1}$, $Q^{-2}PQ^2=P^{1+2^{m-4}\kappa}$, $R^{-1}PR=P^{1+2^{m-4}b_2}$, $R^{-1}QR=QP^{2^{m-4}d_2}$, $R^2=1$, $Q^4=P^4$, $P^{2^{m-3}}=1$. $\kappa=0$; $d_2=0$, $b_2=0$, 1, $b_2=d_2=1$; $\kappa=1$, $d_2=0$, $b_2=0$, 1.

(G') $a=1+2a_1$, $b=2b_1$. The groups for μ and ν odd $\sim$ those for μ and ν even, with $C=\begin{bmatrix} P, & Q, & RP \\ P, & Q, & R \end{bmatrix}$. Transformation of (1), (2), (3), (4), (5), and $Q^4=P^4$ by R, gives $c=2c_1$, $d=1+2d_1$, $\beta_2=0$, and the congruences

$a_1+b_1\equiv c_1+d_1 \pmod{2^{m-5}}$...(10),

$\mu_1\equiv\nu_1 \pmod 2$...(11),

$\mu_1(c_1+d_1)+\nu_1(a_1+b_1)\equiv 0 \pmod{2^{m-5}}$...(12),

$b_1+c_1\equiv 0 \pmod 2$...(13),

$a_1+b_1+c_1+d_1+2[a_1(b_1+c_1+d_1)+b_1^2]+2^{m-5}\kappa(a_2c_1+\mu_1)\equiv 0 \pmod{2^{m-4}}$.(14),

$a_1+b_1+c_1+d_1+2[d_1(a_1+b_1+c_1)+c_1^2]+2^{m-5}\kappa(\nu_1+b_1d_1)\equiv 0 \pmod{2^{m-4}}$ (15),

$[1, y, x]^2=[0, 2\mu_1+y(1+2c_1)+x(1+2a_1)+2y\theta_{x+y}, 2\nu_1+x(1+2b_1)+(-1)^{x+y}y$
$+2d_1y+2^{m-4}(\kappa+\beta_2)\{(x+y)\nu_1+b_1xy\}+2^{m-5}\{\kappa[b_1(x-\theta_x)+c_1(y-\theta_y)+$
$y(x+y)(1+2d_1)+2a_1xy-\theta_{y(x+y)}]+\beta_2[b_1(x-\theta_x)+y\theta_{x+y}(1+2d_1)]\}]$...(16).

The groups for ν_1 other than $0\sim$ those for $\nu_1=0$, through $C=\begin{bmatrix} P, & Q, & R' \\ P, & Q, & R \end{bmatrix}$ where $z''=1$, and y'' and x'' satisfy $2\mu_1+y''(1+2c_1)+x''(1+2a_1)+2y''\theta_{x''+y''}+2\nu_1+x''(1+2b_1)+(-1)^{x''+y''}y''+2d_1y''+2^{m-4}(\kappa+\beta_2)[\nu_1(x''+y'')+b_1x''y'']+2^{m-5}\{\kappa[b_1(x''-\theta_{x''})+c_1(y''-\theta_{y''})+y''(x''+y'')(1+2d_1)+2a_1x''y''-\theta_{y''(x''+y'')}]+\beta_2[b_1(x''-\theta_{x''})+y''\theta_{x''+y''}(1+2d_1)]\}\equiv 2(\mu_1'+\nu_1') \pmod{2^{m-3}}$. The groups for $a_1+b_1\equiv 0 \pmod{2^{m-5}}\sim$ those, where $a_1=b_1=0$, with $C=\begin{bmatrix} P, & Q^{1+2a_1}P^{2b_1}, & R \\ P, & Q, & R \end{bmatrix}$.

(G_1') $b_1=2^{m-6}b_3$, $d_1=-2^{m-6}b_3$; $a_1=c_1=0$.

(G_2') $b_1=-1+2^{m-6}b_3$, $d_1=-2-2^{m-6}b_3$, $a_1=0$, $c_1=1$.

(G_3') $b_1=-2+2^{m-6}b_3$, $d_1=-1-2^{m-6}b_3$, $a_1=1$, $c_1=0$.

(G_4') $b_1=-1+2^{m-6}b_3$, $d_1=-1+2^{m-6}(b_3+2d_3)$, $a_1=c_1=1$.

(G_3') and $(G_4')\sim(G_2')$ and (G_1'), respectively, with $C=\begin{bmatrix} Q^{2y_1}P^x, & Q^{y'}Q^{2x_1'} \\ P, & Q \end{bmatrix}$ where x and y' are odd and the variables satisfy $x_1+y_1-x_1'+y_1'\equiv 0 \pmod{2^{m-5}}$, $a_1'+y_1'\equiv a_1+x_1 \pmod 2$, $x_1+y_1-x_1'-y_1'+a_1-a_1'+b_1-b_1'+2x_1(a_1+b_1-b_1')+2y_1(c_1+d_1-b_1')-2a_1'(x_1'+y_1')+2^{m-5}\kappa\{a_1x_1'+b_1x_1+y_1(1+c_1+b_1')\}\equiv 0 \pmod{2^{m-4}}$, $c_1'+y_1\equiv c_1+x_1' \pmod 2$, $x_1+y_1-x_1'-y_1'+c_1'-c_1+d_1'-d_1+2d_1'(x_1+y_1)+2c_1'(x_1'+y_1')-2y_1'(c_1+d_1)-2x_1'(a_1+b_1)+2^{m-5}\kappa\{c_1'x_1'+d_1'y_1+x_1'(1+b_1)+c_1y_1'\}\equiv 0 \pmod{2^{m-4}}$. The groups in (G_1') and (G_2') are simply isomorphic with the types given below, by $C=\begin{bmatrix} Q, & P, & R \\ P, & Q, & R \end{bmatrix}$, for (G_1) and for

$\kappa=0$ in (G_2); and $C=\begin{bmatrix} P, & QP^{2^{m-4}}, & R \\ P, & Q, & R \end{bmatrix}$, for $\kappa=1$, in (G_2'). There are four types in (G_1') and four in (G_2'). Their defining equations are:

$Q^{-1}PQ=Q^2P^{-1}$, $Q^{-2}PQ^2=P^{1+2^{m-4}\kappa}$, $R^2=1$, $Q^4=P^4$, $P^{2^{m-3}}=1$, $\kappa=0, 1$.

(G_1), $R^{-1}PR=QP^{2^{m-5}b_1}$, $R^{-1}QR=P^{1-2^{m-5}b_1}$; (G_2), $R^{-1}PR=QP^{-2+2^{m-5}b_1}$, $R^{-1}QR=Q^2P^{-1-2^{m-5}b_1}$, $b_1=0, 1$.

§2. $\{P, Q\}$ is of Order 2^m.

$\{P, Q\}=G_m$. $\{P\}=G_{m-3}$. Two cases arise, viz: (A) Q^2 is in G_{m-2}; (B) Q^2 is not in G_{m-2}.

(A) *Q^2 is in G_{m-2}.* Here $Q^{-2}PQ^2=P^{\omega+2^{m-4}\kappa}$...(1), and $Q^4=P^{4\lambda}$...(2). From (1), $[0, -2y_1, x, 0, 2y_1]=[0, 0, (\omega)^{y_1}(1+2^{m-4}\kappa y_1)x]$...(3), and $[0, 2y_1, x]^s=[0, 2sy_1, \{\frac{1+(\omega)^{y_1}}{2}+2^{m-5}\kappa y_1\}(s-\theta_s)x+x\theta_s]$...(4). Let R be some operator in G_{m-1}, not G_{m-2}. Then $G_{m-1}=\{R, G_{m-2}\}$. Also R^2 is in G_{m-2} and in $\{P\}$, for otherwise, $G_{m-1}=\{R, P\}=\{Q', P\}$ which has been considered in §1. Hence $R^2=P^{2\mu}$...(5). In G_{m-1}, $R^{-1}PR=Q^{2a}P^b$...(6), $R^{-1}Q^2R=Q^{2c}P^d$...(7).

(A_1) *$\omega=1$.* From (6) by (4), $b=1+2b_1$. Now (RP^4) and $(RQ^2)^4$ are in $\{P\}$. Hence $a=0$, $c=1$, $d=2d_1$. Then $\{R, P\}$ is of order 2^{m-2} and may be reduced to five cases. $R^{-1}PR=P^{\omega_1+2^{m-4}b_2}$...(6), $R^2=P^{2^{m-4}\mu_1}$...(5). $\omega_1=\pm1$, $b_2=0, 1$, $\mu_1=0$; $\omega_1=-1$, $b_2=0$, $\mu_1=1$. From (6), $[-y, 0, x, y]=[0, 0, (\omega_1)^y(1+2^{m-4}b_2)x]$...(8), and $[y, 0, x]^s=[sy, 0, \{\frac{1+(\omega_1)^y}{2}+2^{m-5}b_2y\}(s-\theta_s)x+x\theta_s]$...(9). Transform (2) and (7) by R. There result $\lambda(1-\omega_1)+d_1\equiv0 \pmod{2^{m-5}}$...(10), $d_1(1+\omega_1)\equiv0 \pmod{2^{m-4}}$...(11). $[1, 2y_1, x]^2=[0, 0, 2^{m-4}\mu_1+4\lambda y_1+2d_1y_1+x\{\omega_1+1+2^{m-4}(\kappa y_1+b_2)\}]$...(12). In G_m, $Q^{-1}PQ=RQ^{2f}P^g$...(13), $Q^{-1}RQ=RQ^{2h}P^n$ or $Q^{2h}P^n$...(14). For the second alternative of (14), the relations are inconsistent. We now transform the defining relations by Q. For $\omega_1=-1$, these results show that the relations are inconsistent. Let $Q'=QP^x$ (x odd and even). There result $d=2^{m-5}d_2$, $g=1+2g_1$, $n=2^{m-4}n_2$, $h=0$ and $\lambda=0$ except where $g_1=-1+2^{m-6}g_2$,—g_2 odd, $f=1$, g_2 odd or even, $f=0$,—when $\lambda=2^{m-6}\lambda_1$, ($\lambda_1=0, 1$); and the congruences

$g_1=2^{m-6}g_2$, $n_2+g_2\equiv\kappa \pmod 2$...(15); and

$g_1=-1+2^{m-6}g_2$, $\kappa f+b_2+n_2+g_2\equiv\kappa \pmod 2$...(16).

$[0, y, 0, z, 0, x]^s=[0, sy+(s-\theta_s)f\theta_{xy}, 0, sz+\frac{s-\theta_s}{2}xy, 0, sx+2^{m-5}s(s-1)x\{b_2xy+\kappa(f\theta_{xy}+y)\}+(s-\theta_s)\{g_1x\theta_y+2^{m-5}[(b_2x+n_2y)z+\kappa x(\frac{y-\theta_y}{2})+y(\frac{x-\theta_x}{2})(b_2+\kappa f+\lambda_1 f)]\}]$...(17).

The groups in (A_1) are simply isomorphic with the types given below, with $C=\begin{bmatrix} P', & Q', & R' \\ P, & Q, & R \end{bmatrix}$, where $P'=Q^{y}R^{z}P^{x}$, $Q'=Q^{y'}R^{z'}P^{x'}$, $R'=Q^{2y_1''}R^{z''}P^{2^{m-5}x_1''}$. In these x and y' are odd; for g_1 even, $x'=2^{m-5}x_1'$; for g_1 odd $y=2y_1$.. The variables satisfy $\kappa(1+f\theta_{x'})+\kappa'(1+g_1\theta_y)+b_2x'\equiv 0 \pmod 2$, $\kappa y_1''+z''(n_2y+b_2)\equiv b_2' \pmod 2$, $f\equiv y_1''+f'(1+f\theta_{x'}) \pmod 2$, $1\equiv z''+f'x' \pmod 2$, $2^{m-5}\{\lambda_1[f(1+x_1)+f'(1+f\theta_{x'})-y_1'']+\kappa[x'y_1+y_1'+f(x'+x_1)]+b_2[x'(1+z)+z'+x_1]+n_2[y(z'-z'')+z]+f'\theta_y[n_2z'+x_1']+g_1'[b_2z+\kappa y_1]\}-2^{m-6}x_1''+x[g_1-g_1'(1+g_1\theta_y)]\equiv 0 \pmod{2^{m-4}}$, $x'(\kappa y_1''+b_2z'')+g_2x_1''+n_2z''\equiv n_2' \pmod 2$, $x'(1+g_1)+2^{m-6}\{\lambda_1(1+fx')-\lambda_1'\}\equiv 0 \pmod{2^{m-5}}$, $\lambda_1y_1''+x_1''\equiv 0 \pmod 2$.

(A_1) $\omega=-1$. The relations when transformed by Q are found to be inconsistent. There are therefore sixteen types in (A), viz:

$Q^{-1}PQ=RQ^{2f}P^{\omega_2+2^{m-5}g_2}$, $Q^{-2}PQ^2=P^{1+2^{m-4}\kappa}$, $R^{-1}PR=P^{1+2^{m-4}b_2}$, $Q^{-1}RQ=RP^{2^{m-4}n_2}$, $R^2=1$, $Q^4=P^{2^{m-4}\lambda_1}$, $P^{2^{m-3}}=1$.

κ	b_2	f	g_2	n_2	λ_1	ω_2	κ	b_2	f	g_2	n_2	λ_1	ω_2	κ	b_2	f	g_2	n_2	λ_1	ω_2
0	0,1	0	0	0	0	1	1	0	0	0	1	0	1	1	0	0	1	0	0	-1
1	1	0	1	0	0		0	0	0	0	0	0,1	-1	0	1	0	0	1	0,1	
1	0	0	1	0	0		1	0	0	0	1	0,1		1	0	1	1	1	0	
0	0	0	1	1	0		0	0	0	1	1	0		0	1	0	1	0	0	

(B) Q^2 is not in G_{m-2}.

$R^{-1}PR=P^{\omega_1+2^{m-4}\kappa}$ ...(1),

$R^2=P^{2^{m-4}\mu}$ ...(2).

From (1), $[-y, 0, x, y]=[0, 0, \{(\omega_1)^y+2^{m-4}\kappa y\}x]$...(3), and

$[y, 0, x]^s=[sy, 0, \{\frac{1+(\omega_1)^y}{2}+2^{m-5}\kappa y\}(s-\theta_s)x+x\theta_s]$...(4).

In G_{m-1}, $Q^{-2}PQ^2=R^aP^b$...(5).

From (5) by (4), $b=1+2b_1$. $(Q^2P)^2=Q^4R^aP^{2b}$. If $a=1$, Q^2P would be an operator Q' of G_{m-1}, where $Q'^2\neq\{P\}$, which was discussed in §1. If $a=0$, $Q^{-2}PQ^2=P^{1+2b_1}$, and Q^2 is in $G'_{m-2}=\{Q_1^2P\}$, which was discussed in (A). Hence there are no new types in (B).

PART 3. THE SQUARE OF EVERY OPERATOR IS IN $\{P\}$.

The group G_{m-1} is determined from the types in §3, Part 2, by replacing m by $m-1$. The types of G_{m-1} are

(A) $Q^{-1}PQ=P^{1+2^{m-4}\kappa}$ ...(1), $R^{-1}PR=P^{\omega_1+2^{m-4}\beta_1}$...(2), $R^{-1}QR=QP^{2^{m-4}b_1}$

...(3), $R^2=1$, $Q^2=1$, $\omega_1=\pm 1$, $\kappa=0$, $\beta_1=0, 1$, $b_1=0$; $\beta_1=0$, $b_1=1$; $\omega_1=-1$, $\kappa=\beta_1=1$, $b_1=0, 1$.

(*B*) $Q^{-1}PQ=P$...(1), $R^{-1}PR=P^{-1}$...(2), $R^{-1}QR=Q$...(3), $R^2=P^{2^{m-h}}$, $Q^2=1$.

Let S be an operator in G_m, not in G_{m-1}.

Then $S^2=P^{2v}$...(4).

In $G_m=[S, G_{m-1}]$, $S^{-1}PS=R^aQ^bP^c$...(5),

$S^{-1}QS=R^eQ^fP^g$...(6),

$S^{-1}RS=R^hQ^iP^j$...(7).

$(SP)^2$, $(SQ)^2$, and $(SR)^2$ are in $[P]$; hence $a=b=e=i=0$, $c=\omega+2^{m-h}c_1$. From (6), $g=2^{m-h}g_1$. Transformation of (7) by S shows $j=2^{m-h}j_1$. In all cases $\nu=0$, except when $c=-1$, when $\nu=2^{m-h}\nu_1$.* The groups in Part 3 are simply isomorphic with the following types:

$Q^{-1}PQ=P$, $R^{-1}PR=P^{\omega_1+2^{m-h}\beta_1}$, $R^{-1}QR=Q$, $S^{-1}PS=P^{\omega+2^{m-h}c_1}$, $S^{-1}QS=QP^{2^{m-h}g_1}$, $S^{-1}RS=RP^{2^{m-h}j_1}$, $S^2=P^{2^{m-h}\nu_1}$, $R^2=1$, $Q^2=1$, $P^{2^{m-3}}=1$.

$\omega=\omega_1=1$, $\beta_1=g_1=\nu_1=0$, $c_1=0$, $j_1=0, 1$; $c_1=1$, $j_1=0$; $\omega_1=1$, $\omega=-1$, $g_1=0$; $\nu_1=0$; $\beta_1=0$, $c_1=0$, $j_1=0, 1$; $\beta_1=0$, $c_1=1$, $j_1=0$; $\beta_1=1$, $c_1=0$, $j_1=0$; $\nu_1=1$, $\beta_1=c_1=j_1=0$; $\omega_1=-1$, $\omega=1$, $\beta_1=0, 1$, $g_1=1$, $c_1=j_1=0$.

*Paragraph 2.

A METHOD OF APPROXIMATION.

By S. A. COREY, Hiteman, Iowa.

The following method of approximation may or may not be new, but as I believe it to be of practical importance I wish to call attention to it, and to point out that to secure the same degree of accuracy, it involves, at least in some cases, less labor than does the method of mechanical quadrature. It can also be used in certain cases where the method of mechanical quadrature fails, because the finite differences used in the latter method cannot always be found between the proper limits. On account of its rapid convergence it can also be used when other common methods fail.

We have the formula,*

$$f(x)=f(0)+\frac{x}{2m}\left\{f(x)+f(0)+2\left[f'\left(\frac{x}{m}\right)+f'\left(\frac{2x}{m}\right)+\ldots+f'\left(\frac{m-1}{m}x\right)\right]\right\}$$

$$-\frac{B_1x^2}{m^2.2!}[f'(x)-f'(0)]+\frac{B_2x^4}{m^4.4!}[f'''(x)-f'''(0)]-\frac{B_3x^6}{m^6.6!}[f^{v}(x)-f^{v}(0)]+\ldots.$$

$$+(-1)^n\frac{B_nx^{2n}}{m^{2n}.(2n)!}[f^{2n}(x)-f^{2n}(0)]+\ldots \quad \ldots(1),$$

(B_1, B_2, B_3, etc., being Bernoulli's numbers, $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, $\frac{1}{30}$, $\frac{5}{66}$, etc.)

For brevity, (1) may be written,

$$f(x)=F(x,m)+\frac{s_2}{m^2}+\frac{s_4}{m^4}+\frac{s_6}{m^6}+\text{etc.} \quad \ldots(2),$$

in which s_2, s_4, s_6, etc., are independent of m. It is evident that m may be considered a convergence factor which makes $F(x, m)$ approach $f(x)$ as m approaches infinity, F and f becoming equal as m becomes infinite. But as m increases, the labor of evaluating $F(x, m)$ also increases, so that, in practice, it is found desirable to obtain one or more of the higher derivatives involved in s_2, s_4, s_6, etc., in order to permit the use of a smaller value of m, and yet to attain the required degree of accuracy. In certain cases it becomes exceedingly difficult to obtain the form and value of the higher derivatives involved in s_2, s_4, s_6, etc., and chiefly for this reason, it becomes desirable to eliminate from (2) certain of these quantities, (s_2, s_4, s_6, etc.), after a certain degree of approximation has been reached. To accomplish this elimination take more than one value of m and set down a separate equation for each value chosen, thus,

*See *Annals of Mathematics*, Second Series, Vol. 5, No. 4, July, 1904.

$$f(x)=F(x, m_1)+\frac{s_2}{m_1^2}+\frac{s_4}{m_1^4}+\frac{s_6}{m_1^6}+\ldots \quad \ldots(3),$$

$$f(x)=F(x, m_2)+\frac{s_2}{m_2^2}+\frac{s_4}{m_2^4}+\frac{s_6}{m_2^6}+\ldots \quad \ldots(4),$$

$$f(x)=F(x, m_3)+\frac{s_2}{m_3^2}+\frac{s_4}{m_3^4}+\frac{s_6}{m_3^6}+\ldots \quad \ldots(5),$$

etc. (In selecting the values of m it is usually convenient to take the highest value such that one or more of the lesser values will be sub-multiples thereof.)

It is evident that from $(r+1)$ such linear equations there may be eliminated r of the quantities s, say s_{2i}, $s_{2(i+1)}$, ..., $s_{2(i+r-1)}$. There results,

$$f(x)=\frac{\begin{vmatrix} M_1 & m_1^{-2i} & m_1^{-2(i+1)} & m_1^{-2(i+2)} & \ldots. & m_1^{-2(i+r-1)} \\ M_2 & m_2^{-2i} & m_2^{-2(i+1)} & m_2^{-2(i+2)} & \ldots. & m_2^{-2(i+r-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ M_{(r+1)} & m_{(r+1)}^{-2i} & m_{(r+1)}^{-2(i+1)} & m_{(r+1)}^{-2(i+2)} & \ldots. & m_{(r+1)}^{-2(i+r-1)} \end{vmatrix}}{\begin{vmatrix} 1 & m_1^{-2i} & m_1^{-2(i+1)} & m_1^{-2(i+2)} & \ldots. & m_1^{-2(i+r-1)} \\ 1 & m_2^{-2i} & m_2^{-2(i+1)} & m_2^{-2(i+2)} & \ldots. & m_2^{-2(i+r-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & m_{(r+1)}^{-2i} & m_{(r+1)}^{-2(i+1)} & m_{(r+1)}^{-2(i+2)} & \ldots. & m_{(r+1)}^{-2(i+r-1)} \end{vmatrix}} \quad \ldots(6),*$$

where M_1, M_2, ..., $M_{(r+1)}$, represent the approximate value of $f(x)$ obtained by (2) for m_1, m_2, ..., $m_{(r+1)}$, respectively, by the use of all the terms preceding s_{2i}/m^{2i}.

If all derivatives higher than the $(2i+2r-1)$th are zero, or if all such *even-numbered* derivatives are equal for $x=x$ and $x=0$, (6) gives the exact value of $f(x)$. In other cases the degree of accuracy attained will be no less than would have been obtained by taking the smallest value of m used in (6) and developing $f(x)$ by (2) so far as to include all the r terms eliminated in (6).

As a special case of (6), let $r=2$, $i=2$. Here $M=F(x, m)+\frac{s_2}{m^2}$, and we get after reducing,

$$f(x)=\frac{m_1^6(m_2^2-m_3^2)M_1+m_2^6(m_3^2-m_1^2)M_2+m_3^6(m_1^2-m_2^2)M_3}{m_1^6(m_2^2-m_3^2)+m_2^6(m_3^2-m_1^2)+m_3^6(m_1^2-m_2^2)} \quad \ldots(7).$$

*Determinants of this form may be readily evaluated. See Weld's *Theory of Determinants*, Articles 23 and 27.

It is clear that the more nearly all the M's approach $f(x)$, the more nearly does (6) give the exact value of $f(x)$, and that this increase in approximation is obtained by sufficiently increasing the integers, m, i, and r, or any of them.

The following points may be noted in passing:

First. After choosing certain values of m, r, and i, $f(x)$ is developed by (6) in linear terms of the $(r+1)$ quantities $(M_1, M_2, \ldots, M_{(r+1)})$, each M being an approximate value of $f(x)$.

Second. If $M=F(x, m)$ and if the values (but not the form) of $f'(x)$ be given at each of the $(m+1)$ equidistant points $(0, x/m, 2x/m, \ldots, x)$ for each of the $(r+1)$ values of m chosen, the value of $f(x)$ may be very closely approximated, although in such case the law of its development would not be expressly stated, but it must be assumed that some such law exists in order that $f(x)$ may be developable in terms of x. This makes the method of practical value in many scientific problems where it becomes necessary to find $f(x)$ approximately, and when the only available data are a number of observations of the values of $f'(x)$ at intervals of x/m, the approximation being closer in this method than in that of mechanical quadrature, especially where the intervals (x/m) are few.

A discussion of the cases where (6) does not give the value of $f(x)$ is not here entered into, but the following considerations will be pertinent:

First. As (6) is obtained from a number of equations, (3), (4), (5), etc., care should be taken that each of these equations actually does give a closer approximation of $f(x)$, when the number of terms employed include those eliminated in deriving (6), than does M.

Second. As (6), (2), and (1) are all dependent on the sum of a number of series developed by Stirling's formula, it becomes necessary that each of these underlying Stirling's series should give a true and convergent development of f between any and all of the adjacent, equidistant points (x/m) at which the value of f' is taken.

The following example has been chosen because its development by formula (1) has been given in the MONTHLY,* Vol. XIII, No. 4, and because its solution by mechanical quadrature has been given by Dr. G. W. Hill in the *Analyst*, Vol. II, p. 120, 1875.†

Example. Evaluate $\int_0^{\frac{1}{2}\pi} \frac{x\,dx}{\sin x(1+.16\cos^2 x)^{\frac{1}{2}}}$.

Taking $r=2$, $i=2$, $m_1=6$, $m_2=3$, $m_3=2$, we get $M_1=1.657{,}626{,}355$, $M_2=1.657{,}490{,}406$, $M_3=1.657{,}013{,}853$. Substituting in (7), we get,

*The value of the integral is incorrectly given in the MONTHLY as, 1.657,636,524, instead of 1.657,636,257. The error is due to the fact that

$$f^{(6)},\ \left[\frac{2596\pi^6}{4200.12^6.6!}\right]$$

is carried out as .000,000,009 instead of .000,000,276.

†See also Hill's *Collected Works*, Vol. I, p. 204.

$$f(\tfrac{1}{2}\pi) - f(0) = \int_0^{\frac{1}{2}\pi} \frac{x\,dx}{\sin x(1+.16\cos^2 x)^{\frac{1}{2}}} = \frac{1728M_3 - 23328M_2 + 233280M_1}{211{,}680}$$

$$= 1.657{,}636{,}33\ldots(8).$$

We may check the accuracy of the work of computation and determine the degree of approximation attained in (8) by finding the values of $f'(\frac{\pi}{8})$ and $f'(\frac{3\pi}{8})$. This done we may take $m_1=6$, $m_3=4$, $m_2=3$. Substituting these values of m_1, m_2, m_3, in (7) we get an expression similar to (8) which gives a value of the definite integral coinciding with the value found in (8) for six decimal places, thus proving the accuracy of the work and showing the number of decimal places to which result found by (8) is correct.

This result is not as accurate as that obtained in the April MONTHLY for the reason that the *smallest* value of m here used is 2. This is therefore a more accurate result than would have been obtained by taking $m=2$ and developing by (1) so far as to include the term involving B_4, but a less accurate result than was obtained in the MONTHLY by taking $m=6$ and developing by (1) far enough to include the term involving B_3. The above result is, however, as accurate as Dr. Hill's and was obtained with much less labor. To obtain a more accurate result than that given in the MONTHLY without finding any higher derivatives than are there given, take $i=4$, $r=1$, $m_1=6$, $m_2=3$, and substitute in (6). By using nine decimals throughout the result is found to be 1.657,636,259. By using ten decimals throughout a result correct to nine or ten decimal places would have been obtained.

APPROXIMATION OF THE GREATEST ROOT OF A CUBIC EQUATION WITH THREE REAL ROOTS.

By CHARLES GILPIN, JR., Philadelphia, Pa.

We are concerned with the "irreducible case," in which Cardan's formula is of no value for computation. By replacing x by $-x$ if necessary, we need consider only the form

$$x^3 - ax - b = 0 \quad (a \text{ and } b \text{ positive}) \quad \ldots(1),$$

the two lesser roots of which are negative and the greatest root, which equals the sum of the lesser ones, is positive. It can be shown* that the greatest root g lies between the limits $\sqrt{a}$ and $\sqrt{(4a/3)}$. From equation (1) we obtain

*$g > \sqrt{a}$ by (2). Since $b^2/4 - a^3/27 < 0$ in the irreducible case, $x^3 - ax - b$ is positive for $x=\sqrt{(4a/3)}$, negative for $x=\sqrt{a}$. EDITOR.

$$x=\sqrt{a+\frac{b}{x}}\ldots(2).$$

It is evident that if we assume for g an approximate value less than the true one, and substitute it in the right hand member of equation (2), the resulting value will be greater than the true value; if we assume for g a value greater than the true one, the resulting value will be less than the true value.

For a first approximation to the value of g, assume any convenient positive value between the limits $\sqrt{a}$ and $\sqrt{(4a/3)}$. For a second approximation, substitute the first approximation in the right hand member of equation (2) and compute the resulting value; etc.

This will give a series of approximations, alternately less and greater than the true value, towards which they converge as a limit.

Example. $x^3-6x-2=0$.

$$\sqrt{6}=2.449,$$

$$\sqrt{6+\frac{2}{2.449}}=2.610,$$

$$\sqrt{6+\frac{2}{2.610}}=2.601,$$

$$\sqrt{6+\frac{2}{2.601}}=2.6019,$$

$$\sqrt{6+\frac{2}{2.6017}}=2.601677,$$

the last being correct to five decimals.

ON THE FORMULA FOR THE AREA OF A CURVE IN POLAR CO-ORDINATES.

By JACOB WESTLUND, Purdue University.

In deriving the formula $A=\frac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2 d\theta$ for the area between a curve and two radii vectores it is customary to consider the area as the limit of the sum of infinitesimal circular sectors. This formula* may, however, be derived directly from the formula $A=\int_{x_1}^{x_2} y dx$ for the area between a curve, the axis of x, and two

*It is very probable that this desirable method of proof occurs in the literature; it has been in use in Professor E. H. Moore's course in Calculus. Ed. D.

ordinates. To do this we assume that the curve may be divided into a finite number of parts such that for any part the abscissa x either increases or decreases constantly when we go from one extremity of the arc to the other. Let us then consider an arc AB, where x and ρ increase when θ increases from θ_1 to θ_2. Let OP be the initial line, A', B' the feet of the perpendiculars on OP from A and B, respectively, and let $OA=\rho_1$, $OB=\rho_2$.

Now we have

$$OAB=OBB'-OAA'-AA'B'B=\tfrac{1}{2}\rho_2{}^2\sin\theta_2\cos\theta_2-\tfrac{1}{2}\rho_1{}^2\sin\theta_1\cos\theta_1-\int_{x_1}^{x_2}ydx.$$

Hence

$$OAB=\tfrac{1}{2}\rho_2{}^2\sin\theta_2\cos\theta_2-\tfrac{1}{2}\rho_1{}^2\sin\theta_1\cos\theta_1-\int_{\theta_1}^{\theta_2}\rho\sin\theta\left(\cos\theta\frac{d\rho}{d\theta}-\rho\sin\theta\right)d\theta$$

since $x=\rho\cos\theta$ increases continuously in the interval (θ_1, θ_2) and has a continuous derivative.

But by integration by parts we have

$$\int_{\theta_1}^{\theta_2}\rho^2\frac{d(\sin\theta\cos\theta)}{d\theta}d\theta=\rho_2{}^2\sin\theta_2\cos\theta_2-\rho_1{}^2\sin\theta_1\cos\theta_1-2\int_{\theta_1}^{\theta_2}\rho\sin\theta\cos\theta\frac{d\rho}{d\theta}d\theta.$$

Hence

$$OAB=\tfrac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2\frac{d(\sin\theta\cos\theta)}{d\theta}d\theta+\int_{\theta_1}^{\theta_2}\rho^2\sin^2\theta\,d\theta$$

$$=\tfrac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2(\cos^2\theta-\sin^2\theta)d\theta+\int_{\theta_1}^{\theta_2}\rho^2\sin^2\theta\,d\theta$$

$$=\tfrac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2d\theta.$$

In exactly the same way we prove the formula for the case when x decreases through the interval (θ_1, θ_2). If we now consider any curve which may be divided into a finite number of parts, each part having the property given above, it is easily seen that the area included between the curve and two radii vectores is given by the formula, $\text{Area}=\tfrac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2d\theta$.

*KINEMATIC GEOMETRY. INVERSION AND INVERSORS.

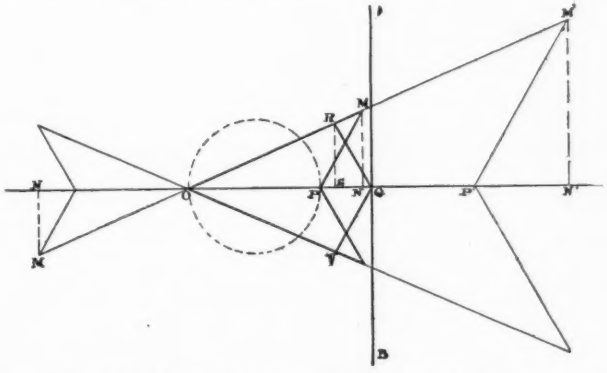
By JOHN JAMES QUINN, Ph. D.

This paper is in the nature of a supplement to the one read by the author before Section A, American Association for the Advancement of Science, in Philadelphia, 1904. In that communication two types of Inversors were exhibited and demonstrated. They represented two distinct groups each embodying the property of inversion, and for special cases they become instruments for describing a line which is mathematically straight.

Further investigation has revealed the fact that an infinite variety of those instruments can be made possessing this principle, differing somewhat in appearance from one another yet essentially the same.

The manner in which they can be constructed is set forth in the following theorems:

THEOREM. *If from any point P in the axis of symmetry OQ of a concave or convex kite lines be drawn parallel to the shorter sides, and terminated by the longer sides (produced if necessary) the product of the distances $OP \times OQ$ is constant, whether the point P be taken within or without the points O and Q.*



GIVEN: The kite $ORQV$; OR produced to M; $MP \parallel QV$; MN and RS perpendicular to OQ.

PROOF. $OP = ON \pm PN = \sqrt{(OM^2 - MN^2)} \pm \sqrt{(MP^2 - MN^2)}$,

$OQ = OS \pm QS = \sqrt{(OR^2 - RS^2)} \pm \sqrt{(RQ^2 - RS^2)}$.

Now $RS = \frac{RO}{MO} . MN$; and $RQ = \frac{RO}{MO} . MP$. Substituting we get

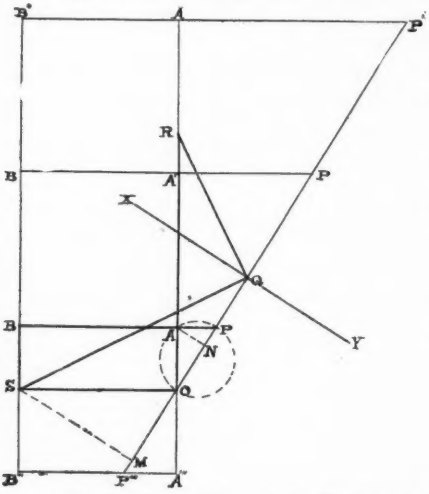
$$OP . OQ = \frac{RO}{MO} [OM^2 \pm MP^2], \text{ a constant.}$$

*Read before Section A, American Association for the Advancement of Science, at the New Orleans meeting, December, 1905.

Therefore the points O, P, and Q are inverse points. Similarly if the points P and Q be interchanged. Q. E. D.

SCHOLIUM. From the above it is evident that if the point O be fixed in position, and the points P, P', etc., be constrained to move in a circle through O, the point Q will describe a straight line, as AB.

THEOREM. *If O and Q be two adjacent vertices of a crossed parallelogram $OQRS$, and P be a point situated on a parallel to OS collinear with O and Q, then $OP \times OQ$ is constant, whether P be within or without the points O and Q.*



GIVEN: The crossed parallelogram $QROS$; $BA \parallel SO$; P on BA produced collinear with O and Q; AN and SM perpendicular to OP.

PROOF. $OP = ON \pm NP = \sqrt{(AO^2 - AN^2)} \pm \sqrt{(AP^2 - AN^2)}$.

$OQ = MQ \pm MO = \sqrt{(QS^2 - SM^2)} \mp \sqrt{(SO^2 - SM^2)}$.

Now $MS = \frac{OS}{AP} . AN$ and $QS = \frac{OS}{AP} . AO$.

Therefore $OP . OQ = \frac{OS}{AP} [AO^2 \mp AP^2]$ is a constant.

Hence the points O, P, and Q are inverse points. Q. E. D.

Similarly, is the points P', P'', etc., be taken.

Evidently then if the point O be fixed and P be constrained to move in a circle through O, the point Q will move in a straight line, as XY. The position of the line described by Q depends upon the position of the center of the circle described by the point P. It is perpendicular to the line connecting O to the center of the circle.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

262. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series $\frac{n}{(4n^2-1)^2}$, beginning with $n=1$, n being always odd.

Solution by G. B. M. ZERR. A. M., Ph. D., Parsons, W. Va.

$$\frac{n}{(4n^2-1)^2}=\tfrac{1}{8}\left(\frac{1}{(2n-1)^2}-\frac{1}{(2n+1)^2}\right).$$

When $n=1, 3, 5, 7, \ldots$

$$\Sigma\frac{n}{(4n^2-1)^2}=\tfrac{1}{8}\left(\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{9^2}+\ldots-\frac{1}{3^2}-\frac{1}{7^2}-\frac{1}{11^2}-\ldots\right)$$

$$=\tfrac{1}{8}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right)-\tfrac{1}{4}\left(\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{11^2}+\ldots\right)$$

$$=\frac{\pi^2}{64}-\tfrac{1}{4}(.1579+)=\frac{\pi^2}{64}-\tfrac{1}{4}\cdot\frac{2\pi^2}{125}\text{ nearly, }=\frac{\pi^2}{64}-\frac{\pi^2}{250}=\frac{93\pi^2}{8000}.$$

We may also write

$$\Sigma\frac{n}{(4n^2-1)^2}=\sum_{m=1}^{\infty}\frac{2m-1}{(16m^2-16m+3)^2}=\tfrac{1}{8}\int_0^1\frac{\tan^{-1}x}{x}dx.$$

This series is discussed by William E. Heal in Vol. IX, pp. 47—49, of the MONTHLY.*

Similar approximations were obtained by S. A. Corey, G. W. Greenwood, and J. Scheffer.

263. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the transcendentals e and π in the form of infinite continued fractions.

*See also an article entitled "Note on the Numerical Transcendants S_n and $s_n=S_n-1$," by Professor W. Woolsey Johnson, in the current *Bulletin of the American Mathematical Society*.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

According to a method due to Euler the series

$$S=\frac{1}{A}-\frac{1}{B}+\frac{1}{C}-\frac{1}{D}+\frac{1}{E}-\ldots$$

may be converted into a continued fraction thus: Putting

$$S_1=\frac{1}{B}-\frac{1}{C}+\frac{1}{D}-\frac{1}{E}+\ldots \quad S_2=\frac{1}{C}-\frac{1}{D}+\frac{1}{E}-\frac{1}{F}+\ldots$$

$$S_3=\frac{1}{D}-\frac{1}{E}+\frac{1}{F}-\frac{1}{G}+\ldots \text{ etc., we get}$$

$$S=\frac{1}{A}-S_1=\frac{1-AS_1}{A};\ \therefore \frac{1}{S}=\frac{A}{1-AS_1}=A+\frac{A^2S_1}{1-AS_1}=A+\frac{A^2}{-A+(1/S_1)},\text{etc.}$$

Thus, $$S=\cfrac{1}{A+\cfrac{A^2}{B-A+\cfrac{B^2}{C-B+\cfrac{C^2}{C-D+\cfrac{D^2}{E-D+\ldots}}}}} \quad \ldots(\text{I}).$$

Since $\frac{1}{4}\pi=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$ we get, substituting in (I), $A=1$, $B=3$, $C=5$, $D=1$, etc.,

$$\frac{1}{4}\pi=\cfrac{1}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\ldots}}}}$$

To convert the series $\frac{1}{a}-\frac{1}{ab}+\frac{1}{abc}-\frac{1}{abcd}+\ldots$ into a continued fraction, we put in (I), $A=a$, $B=ab$, $C=abc$, $D=abcd$, etc., and thus we obtain

$$\frac{1}{a}-\frac{1}{ab}+\frac{1}{abc}-\frac{1}{abcd}+\ldots=\cfrac{1}{a+\cfrac{a}{b-1+\cfrac{b}{c-1+\cfrac{c}{d-1+\ldots}}}} \quad \ldots(\text{II}).$$

To convert $\frac{1}{a}+\frac{1}{ab}+\frac{1}{abc}+\frac{1}{abcd}+\ldots$ into a continued fraction, we have in (II) only to put $-b$, $-c$, $-d$, ... for b, c, d, ... and thus we get

$$\frac{1}{a}+\frac{1}{ab}+\frac{1}{abc}+\frac{1}{abcd}+\ldots=\cfrac{1}{a-\cfrac{a}{b+1-\cfrac{b}{c+1-\cfrac{c}{d+1-\ldots}}}} \quad \ldots(\text{III}).$$

Putting $a=2$, $b=3$, $c=4$, $d=5$, ... we get

$$\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+\ldots=\cfrac{1}{2-\cfrac{2}{4-\cfrac{3}{5-\cfrac{4}{6-\cfrac{5}{7-\ldots}}}}}$$

Hence $e=2+\cfrac{1}{2-\cfrac{2}{4-\cfrac{3}{5-\cfrac{4}{6-\cfrac{5}{7-\ldots}}}}}$

Also solved by G. W. Greenwood, and G. B. M. Zerr.

264. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the invariant $2(a_0a_4-4a_1a_3+3a_2^2)$ of the binary quartic $a_0x_1^4+4a_1x_1^3x_2+6a_2x_1^2x_2^2+4a_3x_1x_2^3+a_4x_2^4$ in terms of roots of the latter.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

It can be shown that, if

$$\begin{aligned}&a_0x_1^4+4a_1x_1^3x_2+6a_2x_1^2x_2^2-4a_3x_1x_2^3+a_4x_2^4\\&\equiv a_0(x_1^2+2px_1x_2+qx_2^2)(x_1^2+2p'x_1x_2+q'x_2^2),\end{aligned}$$

then $4\theta^3-I\theta+J=0$, where $I=a_0a_4-4a_1a_3+3a_2^2$, and $\theta=a_2-a_0pp'$.

Let β, γ be the roots of $x_1^2+2px_1x_2+qx_2^2=0$, and α, δ the roots of $x_1^2+2p'x_1x_2+q'x_2^2=0$. Then

$$\theta=\frac{a_0}{6}\,\Sigma\,\beta\gamma-\frac{a_0}{4}(\beta+\gamma)(\alpha+\delta)=\frac{a_0}{12}(v-w),$$

where $u=(\beta-\gamma)(\alpha-\delta)$, $v=(\gamma-\alpha)(\beta-\delta)$, $w=(\alpha-\beta)(\gamma-\delta)$. The roots of the reduced cubic are therefore,

$$\frac{a_0}{12}(u-v),\ \frac{a_0}{12}(v-w),\ \frac{a_0}{12}(w-u).$$

It is easily found that $u+v+w=0$. Consequently, $\Sigma\,vw=-\frac{1}{2}\,\Sigma\,u^2$, and

$$\frac{I}{4}=-\,\Sigma\,\theta_1\theta_2=-\frac{a_0^2}{144}\,\Sigma\,(uv-uw+vw-v^2)=\frac{a_0^2}{144}\cdot\frac{3}{2}\,\Sigma\,u^2.$$

Hence $I=\frac{a_0^2}{24}\,\Sigma\,u^2$, where u, v, w have the values given above.

Also solved by G. B. M. Zerr.

CALCULUS.

220. Proposed by C. N. SCHMALL, College of the City of New York, New York City.

To determine the least polygon of n sides that can be described about a given circle.

Solution by the PROPOSER.

Let $\phi_1, \phi_2, \ldots, \phi_n$ be the successive angles contained between the lines $l_1, l_2, \ldots$, drawn from the center to the vertices of the polygon, and the radii (r) drawn to the points of contact of the sides. The area of the right triangle whose angle at the center is ϕ_1, will be

$$\triangle = \tfrac{1}{2} r l_1 \sin\phi_1 = \tfrac{1}{2} r . r \sec\phi_1 . \sin\phi_1 = \frac{r^2}{2} \tan\phi_1 .$$

Hence the entire area of the polygon is

$$\Sigma\triangle = \frac{r^2}{2}[\tan\phi_1 + \tan\phi_2 + \ldots + \tan\phi_n].$$

But $\tan\phi_n = \tan[2\pi - (\phi_1 + \phi_2 + \ldots \phi_{n-1})] = \tan[2\pi - \theta]$, where $\theta = \phi_1 + \phi_2 + \ldots + \phi_{n-1}$. Thus $u = \tan\phi_1 + \tan\phi_2 + \ldots + \tan(2\pi - \theta)$ is to be rendered a minimum. Differentiating partially with respect to ϕ_1, we obtain

$$\frac{\partial u}{\partial \phi_1} = \sec^2\phi_1 - \sec^2(2\pi - \theta) = 0.$$

Hence $\phi_1 = 2\pi - \theta = \phi_n$.

In like manner we may show that any angle equals the one preceding it. Hence the minimum polygon is regular.

Also solved by G. W. Greenwood, and J. Scheffer.

DIOPHANTINE ANALYSIS.

136. Proposed by A. H. HOLMES, Brunswick, Maine.

Given $7x^2 - 111 = y^2$. Required a value for y greater than unity which shall be a prime integer.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let $y = q$, $x = p$ be two values that satisfy the equation $y^2 - Nx^2 = -a$, and $y = n$, $x = m$ two values that satisfy the equation $y^2 - Nx^2 = 1$. Then we evidently have $y^2 - Nx^2 = (q^2 - Np^2)(n^2 - Nm^2) = n^2q^2 + N^2m^2p^2 - N(m^2q^2 + n^2p^2) = (nq \pm Npm)^2 - N(mq + np)^2$. Therefore, we can put $y = nq \pm Npm$, $x = mq \pm np$. Substituting numerical values we have, since $y = 8$, $x = 3$ satisfy the equation $y^2 - 7x = 1$,

$$n=\tfrac{1}{2}[(8+3\sqrt{7})^r+(8-3\sqrt{7})^r] \text{ and } m=\frac{1}{2\sqrt{7}}[(8+3\sqrt{7})^r-(8-3\sqrt{7})^r],$$

where for r successive values 1, 2, 3, ... may be put. Since $y=1$, $x=4$ satisfy equation $y^2-7x^2=-111$, we have $q=1$, $p=4$, and thus we find $y=n\pm 28m$, $x=m\pm 4n$. Substituting for r the numbers 2, 3, 4, we get the sets

$$\begin{array}{llllllll} y=76, & y=92, & y=1217, & y=1471, & y=309119, & y=373633, & & \\ x=29, & x=35, & x=460, & x=556, & x=116836, & x=141220, & \text{etc.} & \end{array}$$

Thus $y=1471$ is the least prime which satisfies the equation.

Also solved by A. H. Bell.

GEOMETRY.

290. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Show that the point (1, 1) is a conjugate point on the locus $x^3+y^3-3xy+1=0$.

I. Solution by the PROPOSER.

If a line through the point (1, 1) making an angle θ with Ox have a point P in common with the locus, the coördinates of P, *i. e.*, $1+r\cos\theta$, $1+r\sin\theta$, where r is the distance of P from the point (1, 1), satisfy its equation. Therefore

$$(1+r\cos\theta)^3+(1+r\sin\theta)^3-3(1+r\cos\theta)(1+r\sin\theta)+1=0,$$
$$3r^2(\cos^2\theta+\sin^2\theta+\cos\theta\sin\theta)+r^3(\cos^3\theta+\sin^3\theta)=0.$$

Two values of r are zero, and the point (1, 1) is therefore a double point. But since no real value of θ will make another value of r zero, the point is a conjugate point.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let us take the more general equation $x^3+y^3-3cxy+c^3=0$. Denoting this polynomial by F, we have

$$\frac{\partial F}{\partial x}=3x^2-3cy, \quad \frac{\partial F}{\partial y}=3y^2-3cx.$$

Putting each $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ equal to zero, we get $y=x=c$. It is easy to show that $H=\left(\frac{\partial^2 F}{\partial x\partial y}\right)^2-\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2}=-27c^2$, being negative. Hence (c, c) is a conjugate point.

GROUP THEORY.

8. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

In a chess tournament between eight players, there are seven rounds, the eight players being paired in each round, each pair to be matched once and but once in the tournament. List the possible programs, different except as to notation, *i. e.*, not transformable into each other by a substitution on eight letters. Give the number of conjugate programs of each representative retained.

Note by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

All possible solutions may be transformed into one of the six types given below. For type *F* thanks are given to Dr. Dickson. He kindly sent two solutions for comparison, by which was detected the omission of this type from the final tabulation. The results given have since been carefully checked.

A	15	26	37	48	*B*	15	26	37	48	*C*	15	26	37	48
	16	25	34	78		16	23	45	78		16	23	45	78
	17	24	35	68		17	28	35	46		17	28	35	46
	18	23	45	67		18	25	36	47		18	25	36	47
	12	38	47	56		12	34	58	67		12	34	57	68
	13	28	46	57		13	24	57	68		13	24	58	67
	14	27	36	58		14	27	38	56		14	27	38	56
D	15	26	37	48	*E*	15	26	37	48	*F*	12	38	47	56
	16	23	45	78		16	23	45	78		13	24	58	67
	17	28	35	46		17	24	35	68		14	35	26	78
	18	25	34	67		18	25	34	67		15	46	37	28
	12	36	47	58		12	36	47	58		16	57	48	23
	13	24	57	68		13	28	46	57		17	68	25	34
	14	27	38	56		14	27	38	56		18	27	36	45

Remarks by the PROPOSER.

Dr. Safford has reduced the possible types to *A*—*F*, but has not proved that no two of these are equivalent. I shall prove that this is true, and at the same time answer the remaining questions proposed in the problem.

A very obvious remark shows that *A* is equivalent to no one of the other types. Set $a_1=(15)(26)(37)(48)$, $a_2=(16)(25)(34)(78)$, ... Then $a_1, \ldots, a_7$ together with identity form a commutative group of order 8 generated by a_1, a_2, a_3, with $a_4=a_2a_3$, $a_5=a_1a_2$, $a_6=a_1a_3$, $a_7=a_1a_2a_3$. But the product of the substitutions corresponding to the first and second lines in *B* (or *C*, *D*, *E*) is (1472)(3856), so that we reach no group; similarly for *F*.

We proceed to main problem, that of finding the group of all substitutions*

*For types *A*, *C*, and *D*, Dr. Safford found certain of the substitutions in the course of his reductions, and kindly placed them at my disposal.

leaving a given type unaltered. The order of the group is not the same for any two types, so that no two are equivalent.

For type A, the group is triply transitive, of order 1344, generated by $S=(257)(346)$, $T=(36)(58)$, $V=(1537)(46)$. Obvious products of these replace 1 by 1, ..., or 8. Now S and V transform T into $R=(34)(78)$ and $P=(47)(38)$. Obvious products of these five leave 1 fixed and replace 2 by 2, ..., 8; others leave 2 and 3 fixed and replace 3 by 3, ..., 8. Now $R^{-1}TR=(46)(57)$, which is transformed into (45)(67) by P. Hence there exist substitutions leaving 1, 2, 3 fixed and replacing 4 by 4, 5, 6, or 7, but not by 8, since 8 is necessarily fixed by the fifth line of A. When 1, 2, 3, 4 are fixed, all are fixed. Hence the order is 8.7.6.4.

For type C, the group is simply transitive, of order 24, generated by $R_1=(126)(385)$, $S_1=(267)(345)$, $T_1=(13)(24)(57)(68)$. There exist substitutions, leaving C unaltered, of the form (1) $(2\,x\ldots)$ for $x=2, 6, 7$, but not for $x=3, 4, 5, 8$. When this is verified for $x=3$ and 8, it follows for $x=4, 5$. Thus, if $W=(1)(24\ldots)$ occurred, then would $WS_1^{-1}=(1)(23\ldots)$ occur. Since identity alone leaves 1 and 2 fixed, the order is 8.3.

For type D, the group is simply transitive, of order 96, generated by (1254)(3678), (13)(57), (24)(68), (537)(648), the first transforming the second into (26)(48). Hence there occur substitutions $(1)(2\,x\ldots)$ for $x=2, 4, 6, 8$. I find that no substitution $(1)(23\ldots)$ leaves D unaltered. Hence this is true for $(1)(25\ldots)$ and $(1)(27\ldots)$, in view of the last generator. If, for y even, $(1)(2)(3\,y\ldots)$ occurred, then by transforming by (24)(68), (26)(48) or by their product, we would reach $(1)(32\ldots)$, whereas its inverse does not occur. When 1, 2, and 3 are fixed, all are; hence the order is 8.4.3.

For the type E, the group is transitive, of order 64, and is generated by (1472)(3856), (1876)(2345), (17)(68), (15)(37), (17)(35). Combinations of the first two replace 4 by 1, ..., 8. A substitution which leaves 4 fixed must replace 6 by 6 or 8. Now $(4)(6)(5\,x\ldots)$ occurs if and only if $x=1, 3, 5, 7$. If 1, 4, 6 are fixed, all are. The order is 8.2.4.

For type F, the group is intransitive, of order 42, and generated by (2345678), (346)(587), (38)(47)(56), being the metacyclic group commutative with the cyclic G_7.

For type B it was found that (28)(46) is the only non-identical substitution not altering B and leaving 1 fixed. Another substitution occurring is (15)(37). Hence the group is of order $4n$, $n=1, 2, 3$, or 4. In any event, the order is less than the orders in the previous cases.

The number of conjugate programmes of type A is $8!\div 1344=15$.

MISCELLANEOUS.

158. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

An ingot of pure gold was melted at the Mint and then 10 ounces were taken out and 10 ounces of pure silver added and the contents of the melting pot

mixed thoroughly. This was repeated until there were 10 such operations in all. The contents of the pot being then assayed was found to be nine-tenths fine, or standard gold. What was the weight of the original ingot? There was no loss in the precious metals by the melting.

Solution by the PROPOSER.

Let X=the weight of the original pure gold ingot; n=10=the number of operations; U_n=the weight of pure gold in the pot after the nth operation; c=10=the weight of metal taken from the pot, and also the weight of the silver put in, at each operation; cU_nX^{-1}=the weight of pure gold taken out at the $(n+1)$th operation; U_{n+1}=the weight of the pure gold in the pot after the $(n+1)$th operation; and $U_n=0.9X$.

Equate the elements defined above and we have:

$$U_{n+1}=U_n-cU_nX^{-1}\ldots(1);\text{ or }U_{n+1}=(1-cX^{-1})U_n\ldots(2).$$

This is an equation in Finite Differences. Integrate it, and we have:

$$U_n=C(1-cX^{-1})^n\ldots(3).$$

Equation (3) is true for all values of n. When $n=0$, $U_n=U_0=X$, and $C=X$. Substitute this value of C in (3) and we have:

$$U_n=X(1-cX^{-1})^n\ldots(4).$$

We have $U_n=0.9X$. Eliminate U_n, supply numerical values, reduce, and we have:

$$(1-10X^{-1})^{10}=0.9\ldots(5).$$

Therefore $1-10X^{-1}={}^{10}\sqrt{(0.9)}$; or $X=10\div[1-{}^{10}\sqrt{(0.9)}]$.

The log of $0.9=\bar{1}.954242509=\overline{10}+9.9542425094$, this divided by 10 gives $\bar{1}.9954242509$, which is the log of 0.9895192581, this subtracted from 1, gives 0.01048007419, the log of which is $\bar{2}.0203920257$, which subtracted from 1 (the log of 10) gives 2.9796079743, which is the log of $954.1309293=X$, the original weight of the pure gold ingot, in ounces.

Also solved by S. A. Corey, G. W. Greenwood, and J. Scheffer.

PROBLEMS FOR SOLUTION.

ALGEBRA.

269. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the hyperbolic functions of x in the form of infinite continued fractions.

270. Proposed by C. N. SCHMALL, College of the City of New York, New York City.

Two ferry-boats started simultaneously from opposite sides of a river and one being faster than the other, they met 720 yards from the shore. Each boat remained 10 minutes in its slip to change passengers and started on its return trip, when it was found that they met again 400 yards from the other shore. What is the width of the river?

CALCULUS.

221. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Find $\underset{x \doteq 0}{\text{Lim}} \tan^{-1}x(\log x)$.

222. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

If $s_n = 2\left(\frac{1}{n} - \frac{2}{2n^3} + \frac{1}{5n^5} + \frac{1}{7n^7} - \frac{2}{9n^9} + \frac{1}{11n^{11}} + \ldots\right)$ prove that

$$\log 3 = s_3 + s_4,$$
$$\log 7 = s_2 + s_3 + s_4,$$
$$\log 13 = s_2 + 2s_3 + s_4.$$

223. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Prove that $\underset{n \doteq \infty}{\text{Lim}} \frac{\sum_{\lambda=1}^{n} \lambda^k}{n^{k+1}} = \frac{1}{k+1}$.

224. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Prove that $\int_0^\infty \tan^{-1}(\tan\alpha \sin x)\frac{dx}{x} = \frac{1}{2}\pi \log(\tan\alpha + \sec\alpha)$.

GEOMETRY.

295. Proposed by S. F. NORRIS, Professor of Mathematics, Baltimore City College, Md.

One side and the opposite angle of a triangle are fixed. Find the locus of the center of the inscribed circle. Solve by methods of analytic geometry.

296. Proposed by J. J. QUINN, Ph. D., Warren, Pa.

Given $AB = BC$ perpendicular to each other, and E and M their mid-points, respectively. On AB describe a semi-circle, and draw CE to meet the circumference in D. Draw DM cutting AB in F. In what ratio is AB divided by the point F?

NOTES AND NEWS.

Miss M. M. Young has been appointed instructor in mathematics at Wellesly College.

At Princeton University Dr. R. L. Moore has been appointed instructor in mathematics.

Mr. Ralph M. Barton has been appointed instructor in mathematics at Dartmouth College.

Dr. J. G. Hardy has been promoted to an associate professorship of mathematics at Williams College.

Mr. J. K. Whittemore has been appointed assistant professor of mathematics at Harvard University.

Dr. R. B. Allen of Clark University has been appointed professor of mathematics at Kenyon College, Gambier, Ohio.

Dr. H. F. Stecker has been promoted to an assistant professorship of mathematics at the Pennsylvania State College.

Professor W. B. Smith of Tulane University has been transferred from the chair of mathematics to the chair of philosophy.

Professor J. A. Miller of the University of Indiana has been appointed professor of mathematics and astronomy at Swarthmore College.

Dr. Edward Kasner has been promoted to an adjunct professorship of mathematics in Barnard College, Columbia University.

At McGill University A. S. Eve has been appointed assistant professor of mathematics, and Dr. H. T. Barnes associate professor of physics.

Professor D. E. Smith of Teachers' College, Columbia University, will spend the summer in Spain in seeking for early mathematical manuscripts and text books.

Professor Robert J. Aley of Indiana University was nominated for State superintendent of public instruction of Indiana at the recent Democratic State convention at Indianapolis.

The Randolph Jones Company of Chicago has prepared for sale a large number of Hanstein's model and goniostat for aid in teaching plane and solid geometry, perspective, shadow construction, etc.

Dr. Roxana Vivian, instructor in mathematics at Wellesly College, has obtained a years leave of absence, and will be a member of the faculty of the American College for girls in Constantinople the coming year.

8–9

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIII. AUGUST-SEPTEMBER, 1906. NOS. 8-9.

SOLUTION OF A PROBLEM IN THE THEORY OF NUMBERS.

By E. B. ESCOTT, Chicago, Ill.

PROBLEM.* Let p^n-1 and n have a common divisor d. In what case is $p^{n/\delta}-1$ divisible by $d(p^{n/\delta}-1)$, δ being a divisor of d? This is always true for the particular value $\delta=1$.

Let $p^{n/d}=P$ and $d=\delta a$. Then the problem may be stated as follows: Given that $P^{\delta a}-1$ is divisible by δa, to find the necessary and sufficient conditions that the expression $\dfrac{P^a-1}{\delta a(P-1)}$ shall be integral. [When the latter is integral, it is evident that the condition that $P^{\delta a}-1$ shall be divisible by δa is *a fortiori* satisfied].

Let $a=a_1{}^{a}\, a_2{}^{b}\ldots$, where a_1, a_2, ... are distinct primes. First, suppose that a_1 is a factor of $P-1$. Then $P=1+a^m\kappa$, where $m \geqq 1$ and κ is any integer. Raising both members of this equation to the power a_1, we get

$$P^{a_1}=1+a_1\, a_1{}^{m}\kappa+\ldots\equiv 1(\text{mod } a_1{}^{m+1}),$$

from which we see that $P^{a_1}-1$ is divisible by $a_1{}^{m+1}$ and by no higher power of a_1. Similarly, $P^{a_1{}^{a}}-1\equiv 0 \pmod{a_1{}^{m+a}}$, and therefore P^a-1 is divisible by $a_1{}^{m+a}$ and by no higher power of a_1.

*Question 2932, *L' Intermédiaire des Mathématiciens*, 13 (1906), p. 87, proposed by L. E. Dickson. For application of this question, see an article by L. E. Dickson, *On finite algebras, Goettinger Nachrichten*, July, 1905.